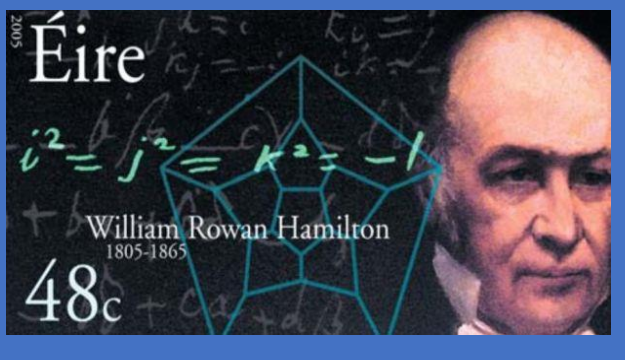
MATHEMATICIANS IN OUR LIVES



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Hamilton supporting teachers

13-14 years olds

SECTION 1 - WILLIAM ROWAN HAMILTON

1. Who is William Rowan Hamilton?

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2. What was his best known discovery?

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3. Where was the bridge that Hamilton carved equations into?

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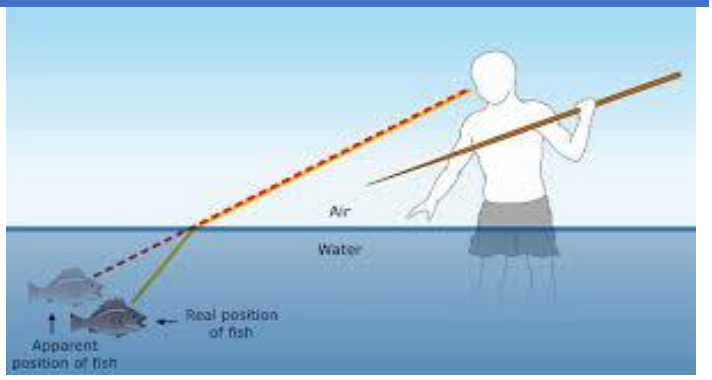
SECTION 2 - OPTICS

4. What is refraction?



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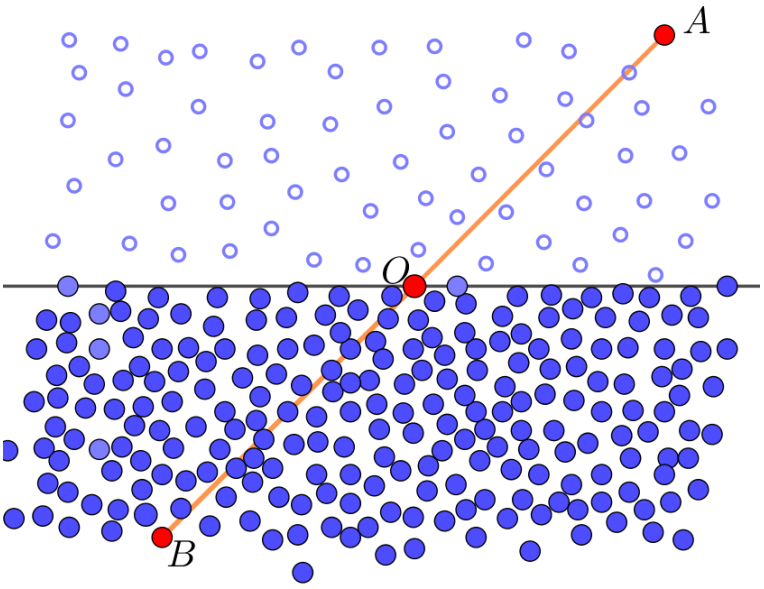
[1]

5. Class Exercise:



Plan Your Trip! Normally, the fastest path between two points is a straight line - but not when you hit obstacles, which cause delays.

In the picture here, count the number of dots that touch the path AB to find how much the traveller is slowed down. The top dots represent gas molecules in the air. The lower dots are water molecules.

Now try to plan a better trip from A to B: Choose a point C on the black separating line, connect it to both A and B by straight line segments, and count the total number of dots you crossed. Is it more or less than on the path AB?

6. What causes refraction to happen?

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7. Why do you think the light travels along the fastest paths?

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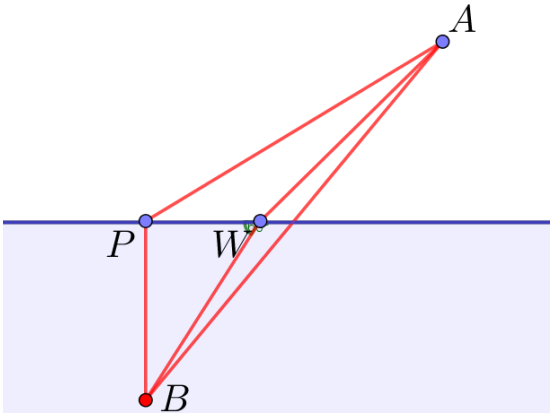
8. In the Light SuperWorld, light rays travel on any paths they like. Three rays called Mr Simples, Mrs Wiseman and James Bold, decide to go from a point A, found 100 meters above water, to a point B, found 100 meters below water. They are warned that travelling through water is slower, namely

   - They can travel at a speed of 300 meters/second through air;
   - But only 225 meters/second through water.

   Mr Simples decides to take a straight line from A to B, a total distance of 255 meters.

   James Bold decides to go as much as possible through air, so he travels 187 m to point P, found exactly above B on the surface of the water, and then from P straight down to B.

   Mrs Wiseman Ray makes some calculation and decides to go about 141.5 meters through the air, heading straight for a point W on the water surface, and then travels about 115.5 meters through the water, from W to B.



   Which Ray gets to the destination fastest? Can you intuitively explain why?

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9. Find some things that Google Maps has in common with Hamilton's approach to optics.

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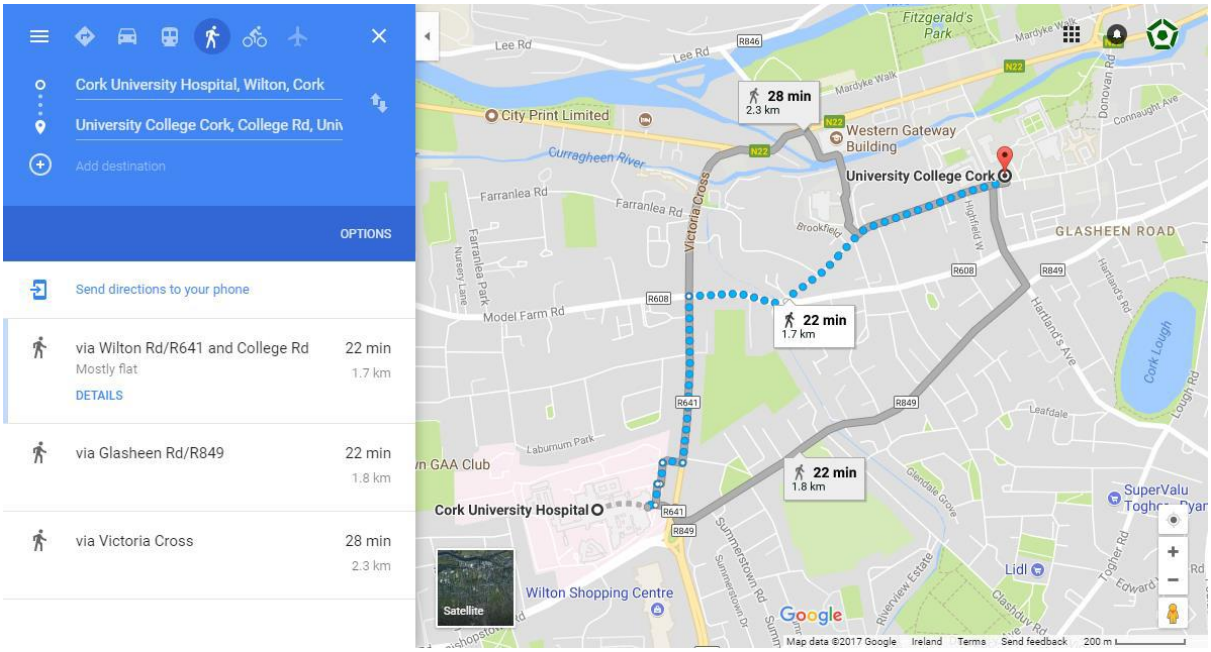

SECTION 3 - GRAPH THEORY

10. What was the game of the Seven Bridges of Königsberg?

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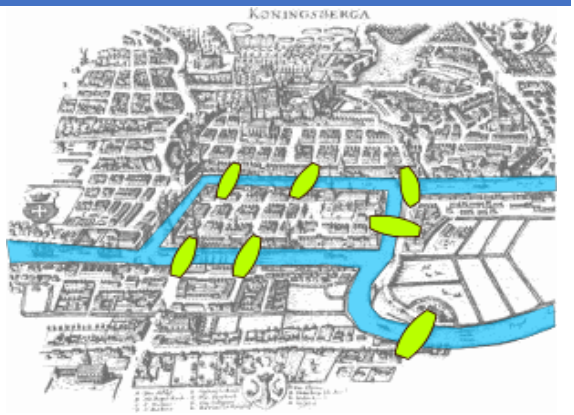

11. Why is it impossible to win the game of the Seven Bridges of Königsberg?

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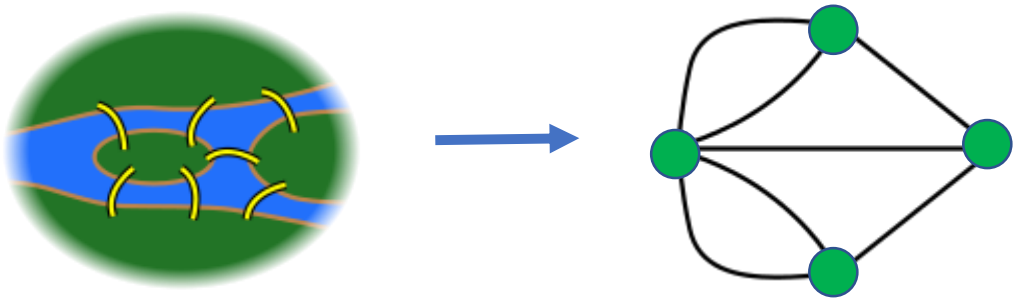
[3]



12. What is a graph in graph theory? Draw an example of a graph.

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13. Below are the 5 Platonic solids and their planar graphs. The graphs are obtained by stretching the bottom face of each Platonic solid to make is much larger than all others, and then squashing the solid from the top until it's flattened and it fits inside its base. The graphs are not in the right order.
    a) What special properties do all the Platonic Solids share?
    b) Connect each graph by an arrow to the Platonic Solid it corresponds to.
    c) Find Hamiltonian circuits (cycles) on each of the 5 graphs.

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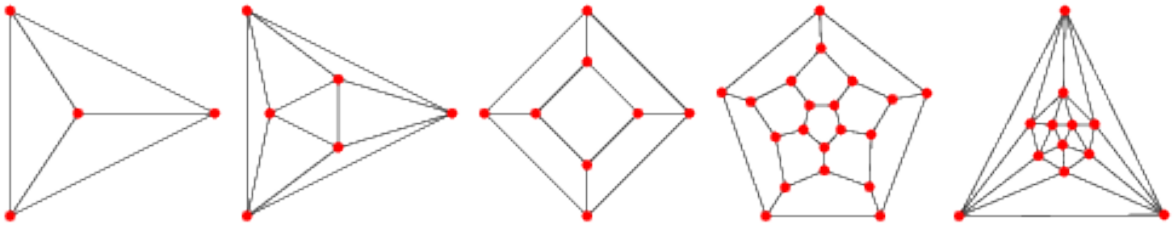
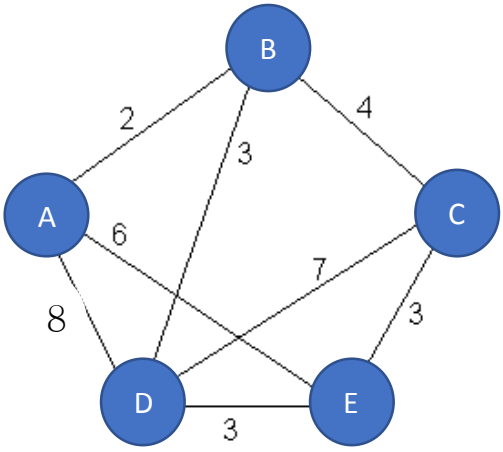


[4]

14. Imagine you are a salesperson who travels around a country selling your product in big cities. Some of the cities are linked by highways, while others aren't, and every highway-link between two cities has an associated distance. You want to visit every city exactly once and finish where you started, while at the same time ensuring you travel the smallest distance possible. Which path do you take?



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SECTION 4 – ALGEBRA AND GEOMETRY

15. Use reflection to explain why multiplying two negative numbers gives you a positive number.

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16. Work out how to reflect around a number other than 0. Let's say that we have a number A on the line and another number x. Write an equation for the reflection of x through A. It should be an expression in x and A.

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17. Show that the two numbers whose sum is 10 and product is 40, would be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ if we allowed for square roots of a negative numbers.

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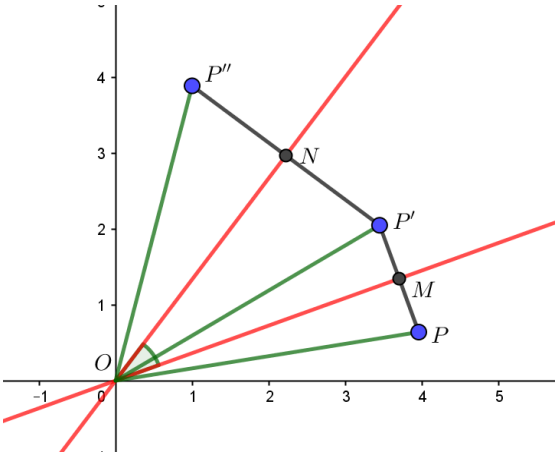
[5]

18. Take two lines l and k meeting at O and with an angle $\alpha = 30°$ between them. Reflect a point P through l and then k successively to get P' and then P''.

If $\angle MON = 30°$ and $\angle POM = 10°$, find $\angle POP''$.



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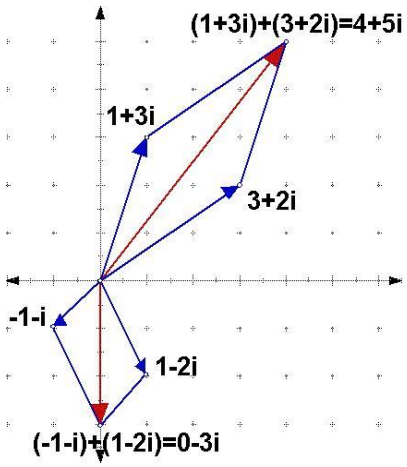


Calculate and draw $(3 + 2i) + (1 - 2i)$ In the diagram.

Do the same for $(3 + 2i) - (1 - 2i)$

20. Fill in the table with the correct Hamilton products. For each box, its row represents the first number in the product, while its column is the second number.

For example, I've placed k in row i and column j because $ij = k$.

×	**1**	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
1				
$\boldsymbol{i}$			k	
$\boldsymbol{j}$				
$\boldsymbol{k}$				

21. Try to multiply these out:

a) $(8\boldsymbol{i})(3 + 4\boldsymbol{i} + 1\boldsymbol{j} + 8\boldsymbol{k}) =$

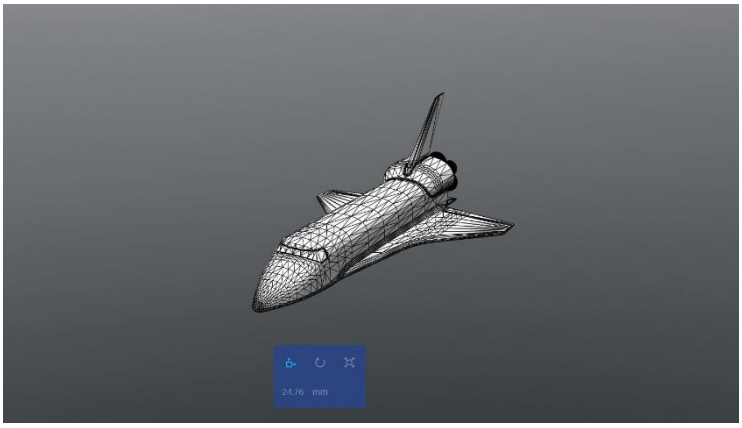
b) $(3 + 4\boldsymbol{i} + 1\boldsymbol{j} + 8\boldsymbol{k})(8\boldsymbol{i}) =$

c) $(3\boldsymbol{j} + 4\boldsymbol{k})(6\boldsymbol{j} + 8\boldsymbol{k}) =$

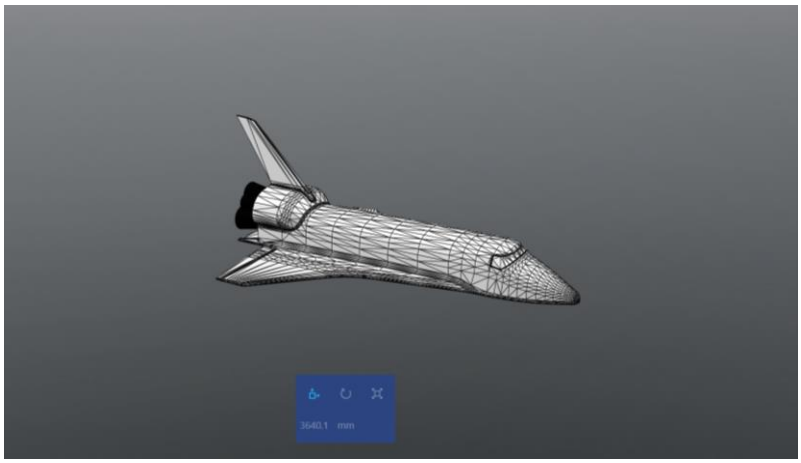
d) $(1 + 3\boldsymbol{i} + 10\boldsymbol{j})(1 + 2\boldsymbol{k}) =$

[6]

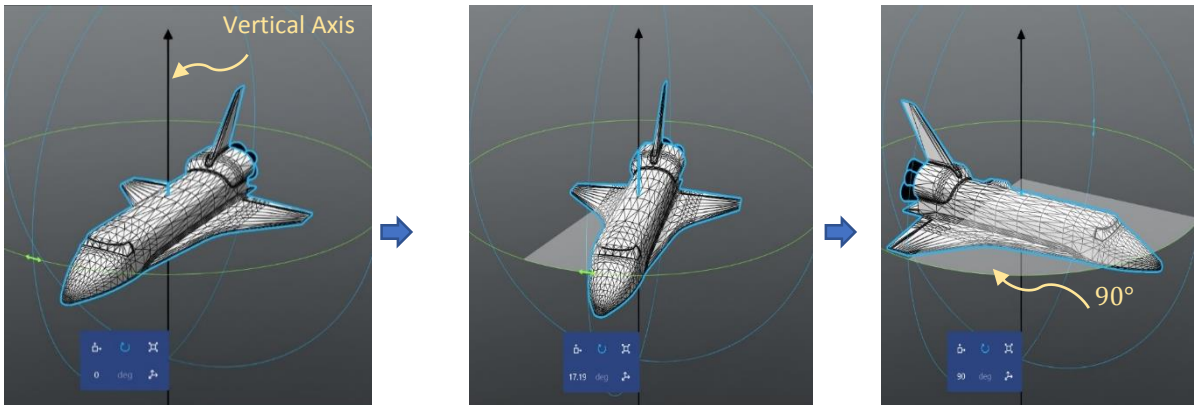
22. Imagine there are astronauts on a spacecraft.



The astronauts are having a tanning competition on board and want to turn the shuttle by $90°$ to its left so that it faces the sun, like this:



How does the shuttle carry out this command (in terms of quaternions)?



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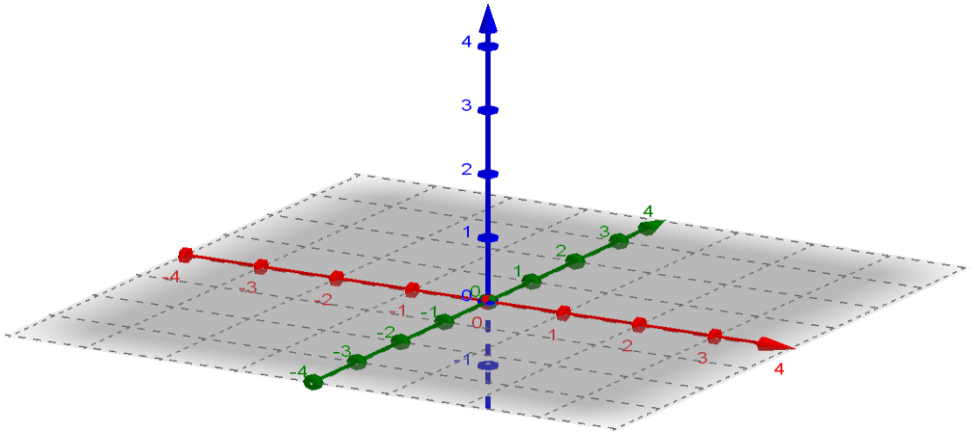
23. Calculate the products iji, iki and iii.

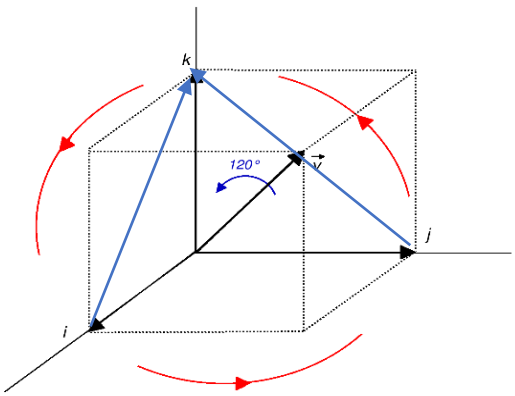
[7]

Now take the number $2i + 3j + k$ and surround it by i and i, like this: $i(2i + 3j + k)i$. Plot your results in 3D space. Try to describe in words what happens to a point $p = ai + bj + ck$ when you surrounded it by i-s.



24. Now try problem 17 again, this time replacing i by j, what do you notice? Now calculate jij, jjj, jkj. What do you notice?

25. Consider the quaternions $u = \frac{1}{\sqrt{2}}(i - k)$ and $v = \frac{1}{\sqrt{2}}(j - k)$. The directions of these vectors are indicated by the blue arrows in the diagram.



Find the result of the transformation

$$p \to upu \to vupuv.$$

in the cases $p = i$, $p = j$ and $p = k$. Use this to explain why this transformation is the rotation by $120°$ around the diagonal of the cube – as shown in the picture.

[8]