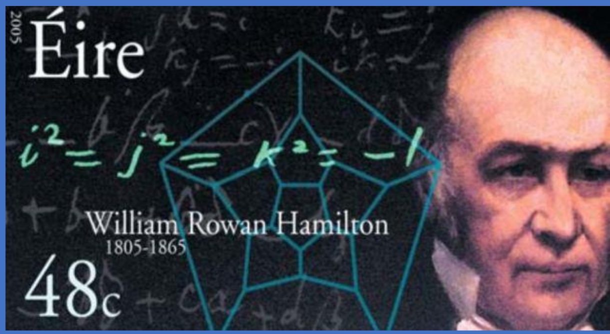


MATHEMATICIANS IN OUR LIVES

With the support of

For 15-16 year olds

INTRODUCTION

As part of the *Mathematicians in Our Lives* programme, the Irish Mathematical Trust (IMT) has developed this package to celebrate the life and legacy of Sir William Rowan Hamilton. The lesson is structured in four sections: After a short review of Hamilton's life and achievements, we will focus on the main areas of his work, with sections on Geometrical Optics, Graph Theory, and Quaternions. Each section discusses the fundamental questions in the area and some of Hamilton's contributions, and includes a set of suggestions for class discussions, games and hands-on exercises. The lesson plan is designed so that you may extract sections to teach or to use the content to build lessons around the information provided. We hope that you enjoy this exploration of the brilliant mind that was Sir William Rowan Hamilton's.

Objectives:

- To introduce William Rowan Hamilton as a person and as a Mathematician.
- To explain the basics of the Laws of Optics, Graph Theory, Quaternions.
- To illustrate the rich interplay between Algebra and Geometry through examples from Optics, Graphs, Complex numbers and Quaternions.
- To solve games, practical tasks and logical exercises on the topics above.
- Advanced students: to deduce the Refraction Law and Rotation Formula for Quaternions.

Required:

- One Quaternion Ball kit, scissors and stapler or sellotape for group of 2-3 students.
- One copy of the worksheet per student.

Lesson time: 1-3 lessons of 40 min each.

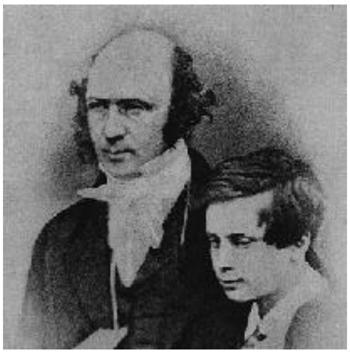
WHO IS WILLIAM ROWAN HAMILTON?

Sir William Rowan Hamilton (1805-1865), one of Ireland's most famous scientists, was a distinguished mathematician, physicist and astronomer. He made a large number of important contributions to Optics, Mechanics, Graph Theory and Algebra. Many notable concepts in physics take their name from him; like the Hamiltonian function and Hamiltonian mechanics, while in Algebra, his best-known discovery is of the Quaternion number system.

HIS LIFE

- *The story of a childhood at the same time normal and exceptional. Invite your students for their opinions on whether there was a connection between the early training in languages and the later proficiency in mathematics?*
- *Starting from Hamilton's story, invite students to discuss how initial challenge and defeat can influence a person's future career.*
- *Ask your students if they ever visited an observatory. Give a short description of one. What connections can be found between a job at an observatory and research in mathematics?*
- *Propose further historical investigation: Compare the lives of George Boole and William Rowan Hamilton. Did they live in Ireland at the same time? Did their lives/work intersect?*

William Rowan Hamilton was born in Dublin on 4th August, 1805. Judging by all his academic exploits at an early age, you wouldn't believe that Hamilton was a healthy boy who loved swimming, nature and jolly gatherings of friends. By the age of 13, nature walks brought out his enthusiasm in the form of poetry in at least 13 languages (Latin, Greek, Persian, Hebrew, Arabic, Sanskrit and others). His education was in the hands of his uncle, an accomplished linguist; Hamilton's mother and father had both died by the time he was 14.



The young Hamilton's first recorded mathematical adventure was a contest that pitted him against another child prodigy, the American "calculating boy" Zerah Colburn (unfortunately, Hamilton lost). Once Hamilton's curiosity about mathematics was ignited, its fire spread rapidly in his imagination. He entered Trinity College Dublin to study both classics and mathematics – achieving the highest honours in both - but he was more and more attracted by the later. He started blending algebra and geometry to study the laws that explain how light moves. He hadn't yet completed his studies when he presented his great work "*Theory of Systems of Rays*" to the Irish Academy (April, 1827). In the same year, before he had time to finish the final exams, he was appointed professor of astronomy, ahead of some well-established astronomers, and despite the fact that he hadn't even applied for the job!

Hamilton worked at the Dunsink Observatory till the end of his life. This was a rich and layered life, with many friends among poets as well as scientists.

HIS WORK – AN OVERVIEW

- *Outline the three main areas of Hamilton's work which will be investigated in this lesson: Optics; Graph Theory; Quaternions.*
- *Discuss practical applications of Hamilton's work: conical refraction; the transition from the Hamiltonian mechanics to quantum mechanics and its uses in modern life; the use of quaternion in describing 3D rotations for airplane/space-ship flights and for computer games. See more resources at the end of this document.*

Hamilton started his scientific work out of curiosity about optics: the laws that explain how light travels through different media like air, water and glass, and how it reacts to obstacles or other changes. In his work "*Theory of Systems of Rays*" and its supplements, he devised the idea of characteristic function: a tool for measuring the time it would take light to travel along various paths, in terms of the start and end coordinates. This allowed him to explain the laws of optics based on the principle that *light always chooses the fastest path* (the *Minimum Principle*).

This brought him to spectacular and unexpected predictions about lights' behaviour. For example, people before him had observed one ray splitting into two or three when passing through a crystal, but Hamilton discovered that in certain cases there would be an infinite number, a cone of refracted rays – which was confirmed by experiments and won him a Royal Medal in Physics.

Even though your laser has emitted just one line beam, from the other side of certain crystals you see it as a ring of light:

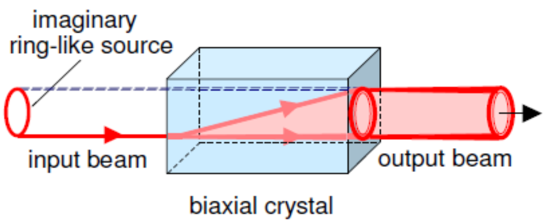
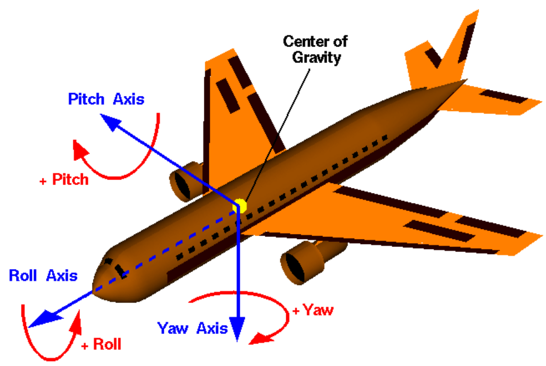


Fig. 1. Transformation of laser beam by conical refraction in a biaxial crystal.

Hamilton's work in geometrical optics fit in well with the new treatment of mechanics developed by J. L. Lagrange (1736-1813), but Hamilton brought a simplicity and clarity which allowed him to carry over all of his methods effortlessly to the most general problems of mechanics.

Almost one hundred years after Hamilton presented his work to the Royal Irish Academy, his methods were found to be just what was needed for the creation of quantum mechanics in 1925-1926, which has in turn brought us the marvels of the digital world.

In his later life Hamilton became more and more intrigued by the interplay between algebra and geometry. This led him to the discovery of the quaternions, a four-dimensional extension of complex numbers determined by the equations $i^2 = j^2 = k^2 = ijk = -1$ (which he famously carved into the side of Broom Bridge in Dublin). He spent the greater part of the rest of his life studying the quaternions and their properties, putting forward applications in the study of rotations that are used in aero- and astronautics to this day.

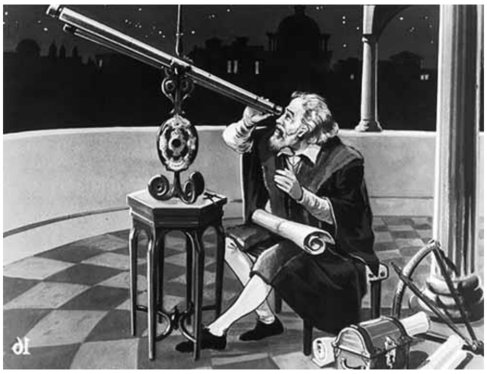


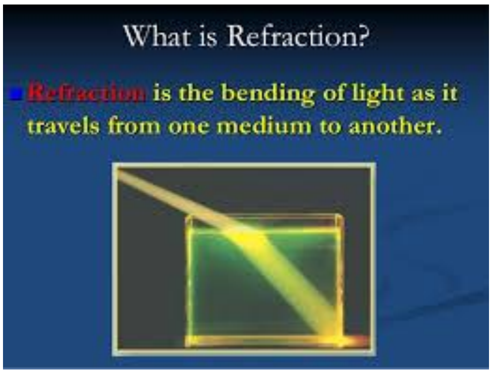
In graph theory he introduced the notions of Hamiltonian paths and circuits while searching for a closed path along the edges of a dodecahedron that visits each vertex exactly once. These ideas generate theorems to this day.

OPTICS

- *A short introduction into Optics through the example of light refraction.*
- *Encourage your students to get involved in a class discussion on the Minimum Principle:*
    - ♦ *How does it apply to Refraction in Optics?*
    - ♦ *Why do they think the principle holds true?*
    - ♦ *How did Hamilton apply it and how does it relate with the Google Maps directions app?*
- *You may organize the students in teams of 2-3 and let them work on choosing the best paths between two points when the medium of propagation changes.*

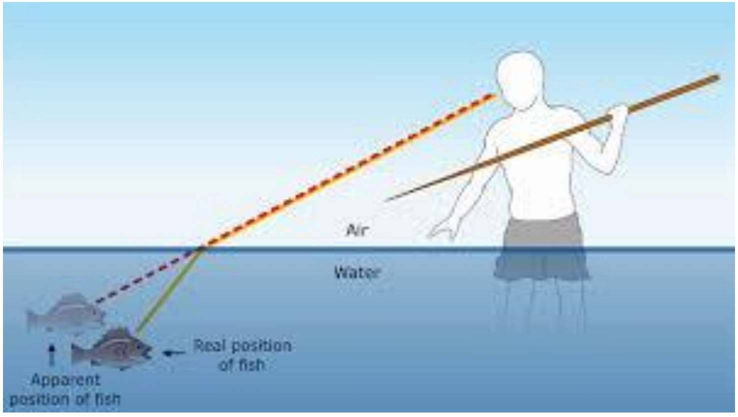
For millennia, people have been attracted to the night sky and the movement of stars – the main questions of astronomy. This is how Galileo Galilei (1564-1642) had come to invent the telescope in 1609, by cunningly exploiting a property of light called refraction.





Since then, the best minds of their time tried to find the true explanation for the refraction of light.

For example, when passing from water to air, the light ray bends. This is why our minds get tricked into perceiving a fish as closer to the surface than it is. Indeed, on the way from the fish to our eye, the light ray had bent, but our minds still thinks it's straight. So in our mind we "see" the fish in an imaginary position along a straight line, instead of its real position lower down.



REFRACTION AND THE MINIMUM PRINCIPLE

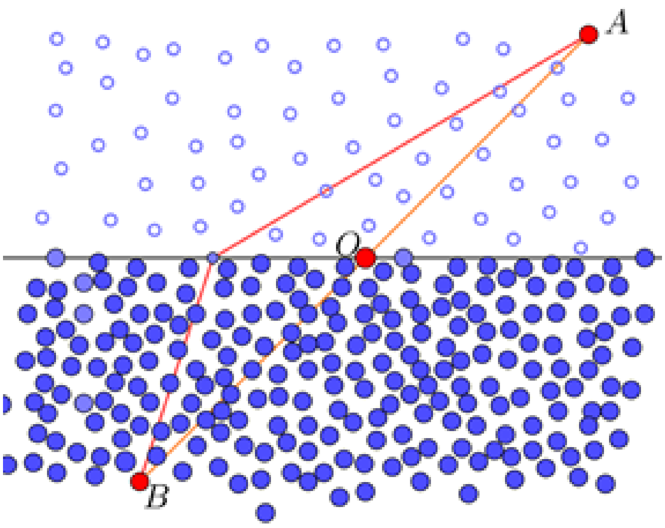
Class Discussion: So, what causes the light to bend when passing from water to air?

Answer: The light travels faster through air than through water.

Class Discussion: Why do you think the light travels faster through air than through water?

Answer: If we could look at air and water through a powerful microscope, we'd see that they are made of little pieces we call molecules. The air is made of molecules of different gases, which keep apart from each other. The water molecules are more crowded. When the light hits a molecule, this shoves the light out of its path a little. Imagine getting shoves every other step – this is bound to slow you down.

Example: Normally, the fastest path between two points is a straight line - but not when you hit obstacles, which cause delays.



Explain why the longer red path in this figure could be faster than the straight path AOB.

Answer: The red bent path is an example of a faster trip:

AOB hits 13 dots.

The red bent path hits about 8 dots. It is chosen so as to make the trip through water shorter, without making the trip through air too long.

Inspired by the ancient work of Heron of Alexandria (c. 10-70 A.D.), the French mathematician Pierre Fermat (1601-1665) came up with a *Minimum Principle*, which basically states:

> *The light ray always travels along the fastest path.*

Class Discussion: How does a light ray plan its journey?

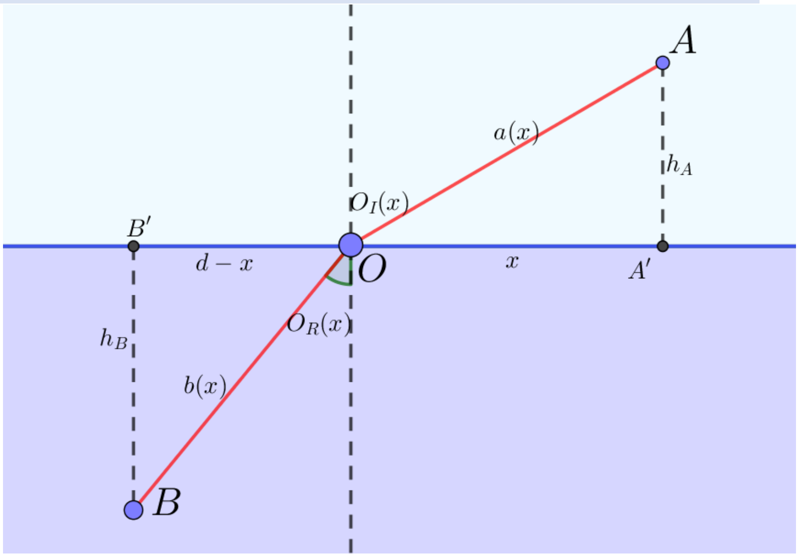
If you don't suppose that Light is as thoughtful as a human being, you might think there is something funny with the idea that the Light knows where it wants to go and plans the whole trajectory in advance.

What's going on? How does the *Minimum Principle* make sense?

Answer: First off, distances; when Hamilton and others applied the Minimum Principle, they were thinking of light travelling extremely small distances, so that you might argue the planning wouldn't require that much foresight. Secondly, we might think of the *Minimum Principle* more as an example of *observer effect*: we think that that's how light behaves because this behaviour is the only one that we can observe. Consider this: a light source usually sends rays in many directions. However, the light rays that start at A in a "wrong" direction will meet more and more obstacles that jolt and deflect them further away from the target. Thus their chances of hitting the target B (where we patiently await) become so small that we can hardly notice any such rays reaching the destination. In fact, our eye is not trained to notice such small effects.

On the other hand, finer measuring instruments may detect "stray" rays, with various frequencies. This kind of probabilistic thinking applied at extremely small scales inspired Quantum Mechanics, an area of Physics to which we owe much of our understanding of nature at atomic level, as well as semiconductor-based electronics, (computers, smartphones), optical cable telecommunication (the Internet), and other features of modern life.

THE REFRACTION FORMULA



When a light ray passes the border $A'B'$ between two mediums, it changes the angle it forms with the vertical line.

The angle O_I formed by the ray with the vertical at the point of intersection with the boundary is called the angle of incidence and the angle O_R made with the vertical after refraction is called the angle of refraction, as described in pictures. Let v_I be the speed of light above the border, and v_R be the speed of light below the border. *The Refraction Formula* tells us exactly where the light crosses the border:

$$\frac{\sin O_I}{\sin O_R} = \frac{v_I}{v_R}$$

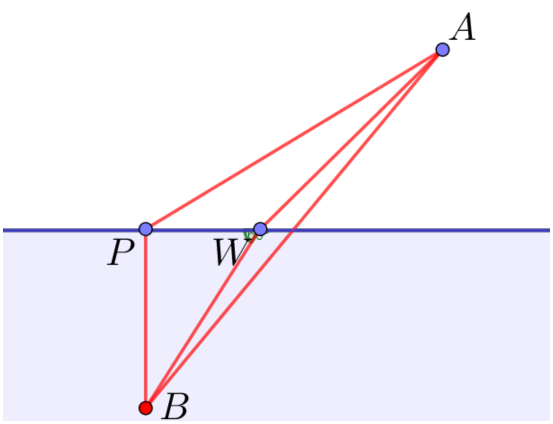
Play hands-on with refraction angles here:

www.physicsclassroom.com/Physics-Interactives/Refraction-and-Lenses/Refraction/Refraction-Interactive

Exercise (Ordinary Level): How to find the perfect path – with numbers.

In the Light Super-World, light rays travel on any paths they like. Three rays called Mr Simples, Mrs Wiseman and James Bold, decide to go from a point A, found 100 meters above water, to a point B, found 100 meters below water. They are warned that travelling through water is slower, namely

- They can travel at a speed of 300 meters/second through air;
- But only 225 meters/second through water.



Mr Simples decides to take a straight line from A to B, a total distance of 255 meters.

James Bold decides to go as much as possible through air, so he travels 187 m to point P, found exactly above B on the surface of the water, and then from P straight down to B.

Mrs Wiseman Ray makes some calculation and decides to go about 141.5 meters through the air, heading straight for a point W on the water surface, and then travels about 115.5 meters through the water, from W to B.

Which Ray gets to the destination fastest? Can you intuitively explain why?

Answer: Using $time = distance/speed$ separately through air and water, we get:

For Mr Simples: $t_{air} + t_{water} = \frac{127.5}{300} + \frac{127.5}{225} = 0.992\ s.$

For James Bold: $t_{air} + t_{water} = 187/300 + 100/225 = 0.623 + 0.444 = 1.068\ s.$

For Mrs Wiseman: $t_{air} + t_{water} = 141.5/300 + 115.5/225 = 0.47167 + 0.51333 = 0.985\ s.$

As light is slower through water, it makes sense to try to shorten the distance travelled through water, even if this makes the trip through air a little longer, too. This is the strategy that both Mrs Wiseman and James Bold took. However, James Bold lengthened the total distance too much. Whereas Mrs Wiseman tried to find a balance between the time spent in the air and the time spent in the water.

Problem (Higher Level): Where does the Refraction Formula come from?

Based on the minimum principle, we can show that the fastest path happens exactly when the Refraction Formula holds. We will use the Refraction Formula in its form $\frac{\sin O_I}{v_I} - \frac{\sin O_R}{v_R} = 0.$

Fix a source *A* and a target *B.* Imagine sliding the point *O* along the border, so that the distance $x = |OA'|$ varies in between *0* and d =$|A'B'|$. Notice all the quantities that change with x. They are denoted as functions of x in the picture*:* $a(x)$, $b(x)$, $O_{I(x)}$, $O_R(x)$, $d - x$ as well as $t(x)$, the total time for the trip AOB.

Choose two points inside the segment $A'B'$ such that $x_1 > x_2$.

(a) Check that the following equations are true:

$$\sin O_I(x_1) = \frac{x_1}{a(x_1)} > \frac{x_1 + x_2}{a(x_1) + a(x_2)} > \frac{x_2}{a(x_2)} = \sin O_I(x_2)\ ,$$

$$-\sin O_R(x_1) = -\frac{d - x_1}{b(x_1)} > -\frac{2d - x_1 - x_2}{b(x_1) + b(x_2)} > -\frac{d - x_2}{b(x_2)} = -\sin O_R(x_2)\,,$$

(b) To compare $t(x_1)$ and $t(x_2)$, calculate the difference $t(x_1) - t(x_2)$ and show that

$$\frac{t(x_1) - t(x_2)}{x_1 - x_2} < \left(\frac{\sin O_I(x_1)}{v_I} - \frac{\sin O_R(x_1)}{v_R}\right).$$

Hence, if $\frac{\sin O_I(x_1)}{v_I} - \frac{\sin O_R(x_1)}{v_R} = 0$ (Refraction Formula), we have $t(x_1) < t(x_2)$ (The time of the trip is shorter if you pass through the point which satisfies the Refraction Formula).

Hint: Use the difference of two squares formula in the form $a(x_1) - a(x_2) = \frac{a^2(x_1) - a^2(x_2)}{a(x_1) + a(x_2)}$.

(c) Similarly, if $x_2 > x_1$, you can check $\left(\frac{\sin O_I(x_1)}{v_I} - \frac{\sin O_R(x_1)}{v_R}\right) < \frac{t(x_2) - t(x_1)}{x_2 - x_1}$.

Solution: (a) After multiplying through and cancelling terms, the inequalities reduce to

$\frac{x_1}{a(x_1)} > \frac{x_2}{a(x_2)}$. Since all terms are positive, we may square: $\frac{x_1^2}{a^2(x_1)} > \frac{x_2^2}{a^2(x_2)}$.

Using $a^2(x) = x^2 + h_A^2$, multiplying through and simplifying terms we reduce to $x_1^2 > x_2^2$.

The second part is the same type of calculation but for the quantities below the border line $A'B'$.

(b) The time of the trip AOB is $t(x) = \frac{a(x)}{v_I} + \frac{b(x)}{v_R}$.

$$t(x_1) - t(x_2) = \frac{a(x_1) - a(x_2)}{v_I} + \frac{b(x_1) - b(x_2)}{v_R} = \frac{a^2(x_1) - a^2(x_2)}{v_I[a(x_1) + a(x_2)]} + \frac{b^2(x_1) - b^2(x_2)}{v_R[b(x_1) + b(x_2)]} =$$

$$= \frac{x_1^2 + h_A^2 - x_2^2 - h_A^2}{v_I[a(x_1) + a(x_2)]} + \frac{(d - x_1)^2 + h_B^2 - (d - x_2)^2 - h_B^2}{v_R[b(x_1) + b(x_2)]}$$

$$= (x_1 - x_2)\left(\frac{x_1 + x_2}{v_I[a(x_1) + a(x_2)]} - \frac{2d - x_1 - x_2}{v_R[b(x_1) + b(x_2)]}\right)$$

$$< \ (x_1 - x_2)\left(\frac{\sin O_I(x_1)}{v_I} - \frac{\sin O_R(x_1)}{v_R}\right)$$

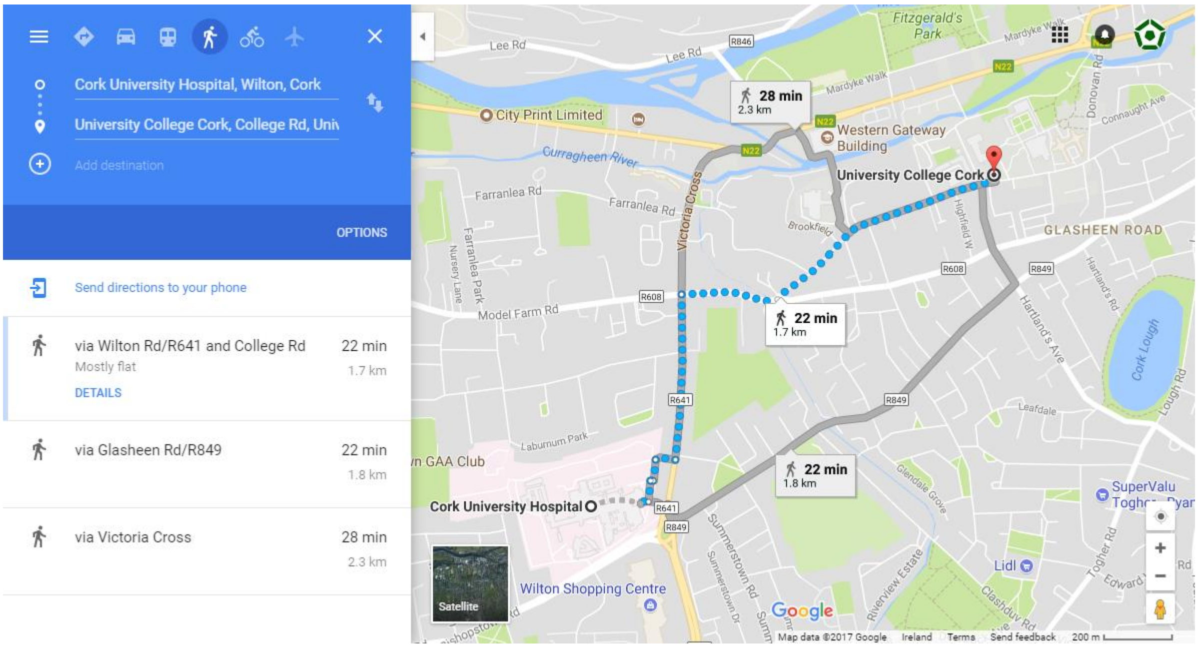
(We used (a) to reduce to the last line). Thus to get $t(x_1) < t(x_2)$ for all $x_2 < x_1$, we need $\frac{\sin O_I(x_1)}{v_I} = \frac{\sin O_R(x_1)}{v_R}$ which is the Refraction Formula. A similar reasoning works in the case of $x_2 > x_1$, but then we would need to use the other side of inequality (a), (after swapping indices 1 and 2).

SO… WHAT ABOUT HAMILTON?

Hamilton's great insight was a smart way to calculate how fast paths are. He designed his calculation as a function of the coordinates of both the starting point and the target. In the previous example, those would be (x, h_A) and $-(d - x, h_B)$. In many more complicated problems, this viewpoint brought clarity and simplifications.

Today, Google Maps works much like Hamilton's characteristic functions:

- It can take as inputs your starting points and the desired destination
- It calculates the duration of each possible path
- And it highlights the fastest path for your convenience:



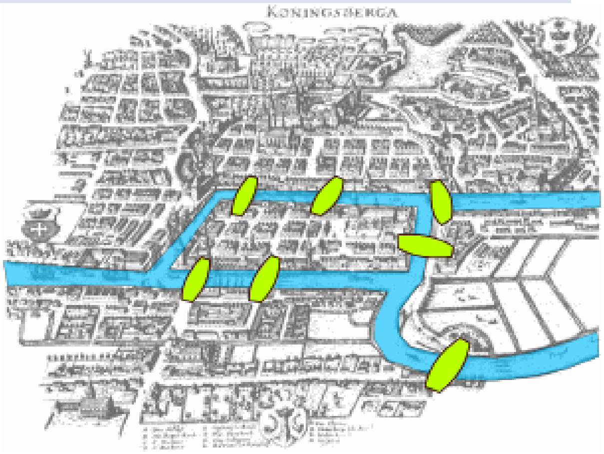
Google Maps' is a simpler set-up than in Optics, because cities have a finite numbers of possible paths. A map can be modelled mathematically by a graph whose points are all addresses and edges are the streets between them. You might not be surprised to find that Hamilton was also interested in graphs and their properties, and he has a special type of graphs named in his honour!

But wait! Guess what? All the data used by Google Maps in its algorithms was gathered by the Global Positioning System, a network of satellites around the Earth. To determine distances, they use atomic clocks designed using principles of Quantum Mechanics – which profited much from Hamilton’s mathematical formulations.

GRAPH THEORY

- *Introduce the area of Graph Theory using a famous puzzle.*
- *Define the notions of Hamiltonian paths, cycles and graphs.*
- *Play some games based on identifying Hamiltonian cycles, and work through some applications.*

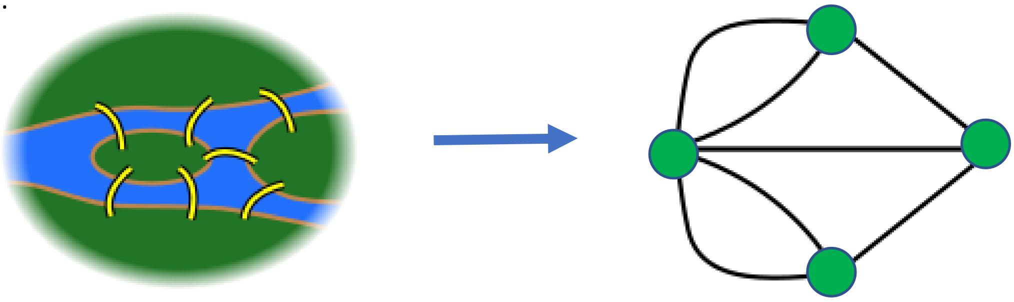
THE SEVEN BRIDGES OF KÖNIGSBERG



The oldest and most famous use of graphs to describe travel around cities comes from the city of Königsberg (now the Russian city Kaliningrad). During the time of the Swiss mathematician Leonhard Euler (1707 – 1783), this was a Prussian city that lay on the Pregel River. A small island was located in the middle of the river at the city centre, and the 4 separate land masses were joined by seven bridges as shown.

The story goes that the people of the city invented a game, whereby they had to try to find a route through the city centre that crossed each of the seven bridges exactly once (without necessarily starting and finishing at the same point). Of course, going half-way across a bridge and turning back was not allowed, and neither was swimming, jumping the gap or running down the bank to look for an eighth bridge or hovercraft. Provided these rules were obeyed, it seemed that no-one could find a solution. Can you? Give it a go!

….but don’t spend too long at it, because it’s actually impossible. In fact, Leonhard Euler proved mathematically that no solution exists, and in doing so kick-started graph theory. Euler discarded most of the beautiful features of the 4 land areas in the city, and represented each area by one node (dot). He could then focus on the bridges and represented them as edges (curved lines connecting the dots:

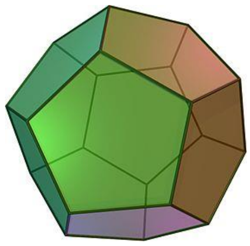


He didn't care about correct sizes and locations, all he was interested in is what connects to what. Can you see the connection between the two pictures above? Next, Euler noticed that, except for the start and end points of your trip, whenever you enter a node by an edge, you also leave it by an edge, i.e. edges are used in pairs, and as a result, there must be an even number of edges connected to each vertex that isn't the start or end point. But you'll notice that all of the nodes here have an odd number of edges. Because of this, Euler concluded that you can't win the game.

As well as being a fun puzzle, this led to the mathematical field of graph theory. For the first time, a real-world situation had been replaced by an equivalent *graph*, the problem had been solved in this abstract setting, and the result translated back into the real world. A *graph* in graph theory is just a number of nodes connected by edges. We're now going to see how Hamilton contributed to the study of graphs.

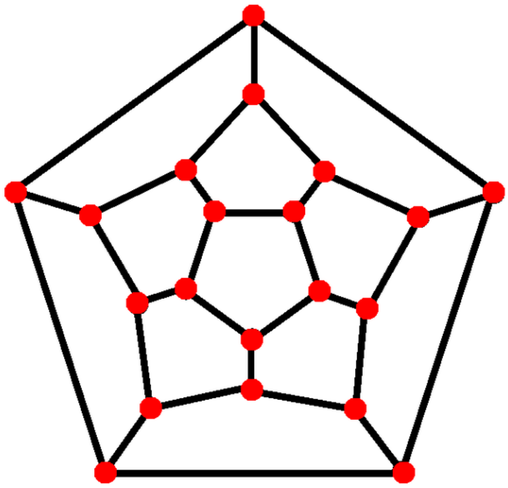
HAMILTONIAN PATHS

What we were looking for in the last section was an *Eulerian Path*, or a path through a graph that visits each edge exactly once. Hamilton was fascinated by shapes like the dodecahedron (aka football) and he started searching for a way along the edges that visits each vertex (node) exactly once. He didn't care about using all edges.



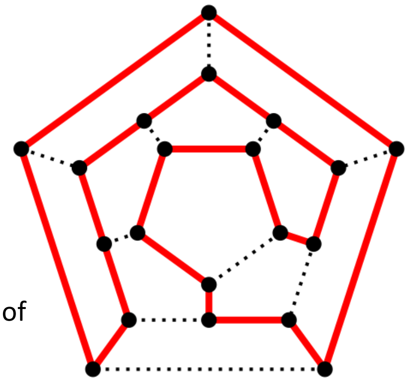
A path that visits each node of a graph once is now called a *Hamiltonian Path*, while a *Hamiltonian Cycle* is a Hamiltonian path that starts and finishes at the same point. The task of finding a Hamiltonian cycle on the edge-graph of a regular dodecahedron is called Hamilton's game or the icosian game. Let's give it a go!

First of all, let's change it to a 2-dimensional problem. Like in the last section, we don't care about the distances between nodes, provided the same things are connected to each other. So, we can "flatten out" the dodecahedron into a 2-D graph:



Exercise: Find a Hamiltonian cycle in this graph:

Sample solution:



Hamilton invented a new mathematical method called icosian calculus and tried to make this into a commercial product. However this ended in failure because the number of solutions people could find was small enough and they became bored of it too quickly.

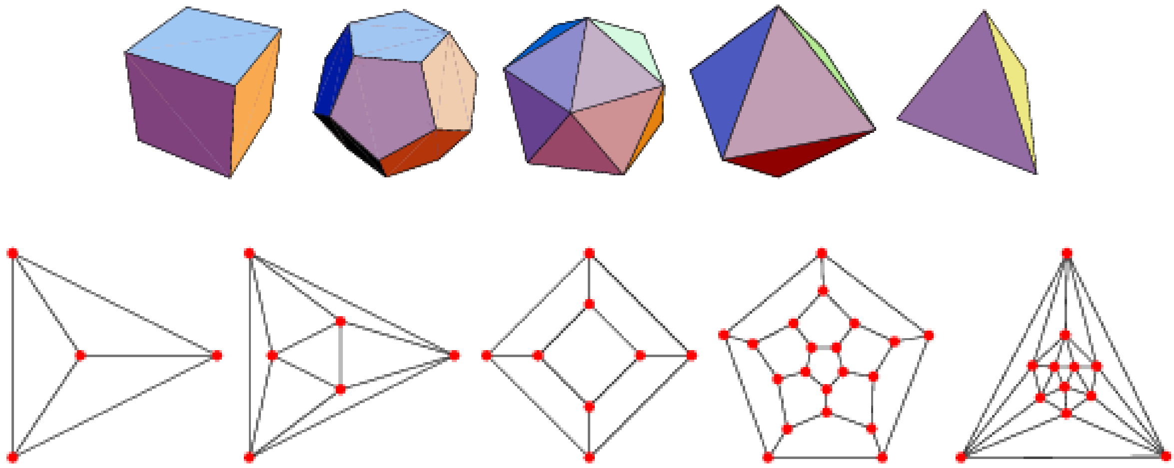
The notion of Hamiltonian paths and circuits is the most interesting aspect of this story, and is an important part of graph theory to this day.

HAMILTONIAN GRAPHS

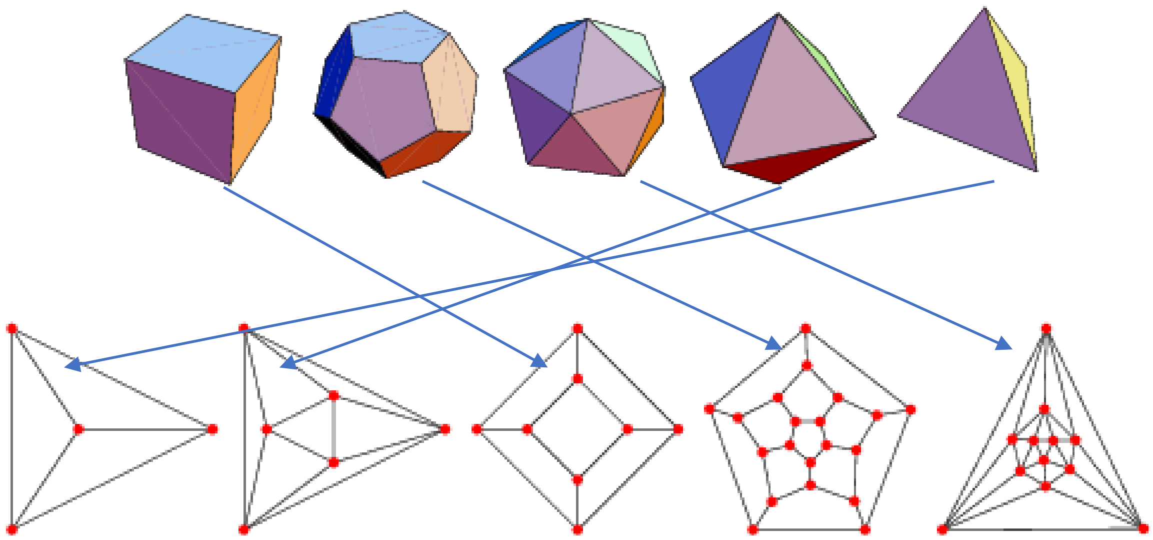
A graph is called Hamiltonian if it has at least one Hamiltonian Cycle in it. Often, graph theorists are just concerned with whether or not a graph is Hamiltonian, and not the finding of a Hamiltonian circuit in the graph. There are some really interesting theorems that can tell, based on the number of nodes and edges, whether or not a graph is Hamiltonian.

Exercise: Below are the 5 Platonic solids and their planar graphs. The graphs are obtained by stretching the bottom face of each Platonic solid to make is much larger than all others, and then squashing the solid from the top until it's flattened and it fits inside its base. The graphs are not in the right order.

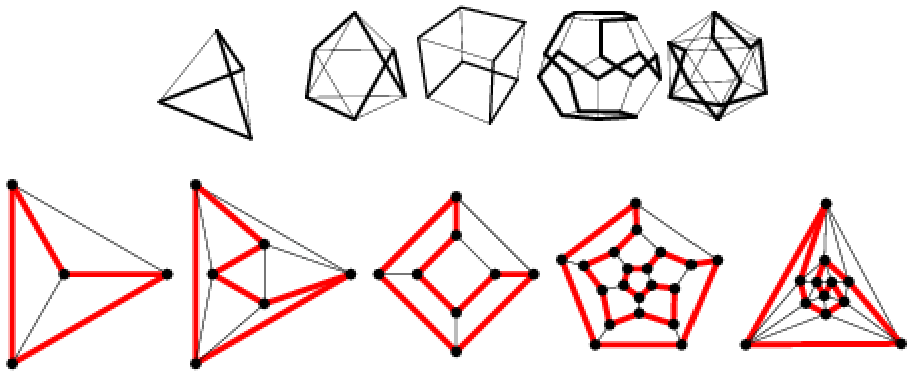
a) What special properties do all the Platonic Solids share?
b) Connect each graph by an arrow to the Platonic Solid it corresponds to.
c) Find Hamiltonian circuits (cycles) on each of the 5 graphs.



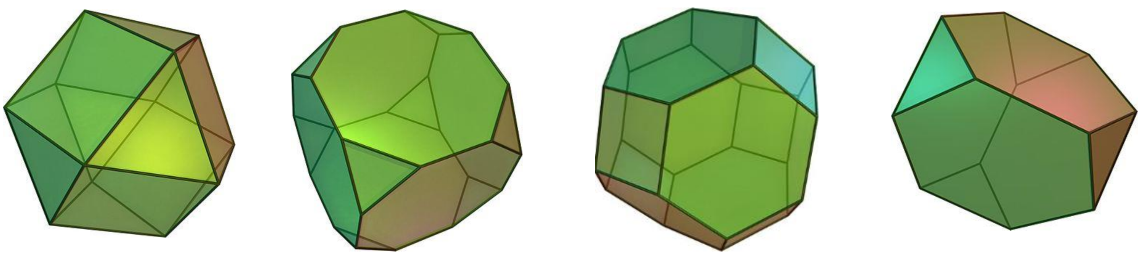
Solution: All faces of a Platonic solid look exactly the same in shape and size. All vertices look the same in number of edges connected to them and the angles around them. In fact, there are no other such shapes with the same properties.



Here are the Hamiltonian graphs (from Wolfram Mathworld):



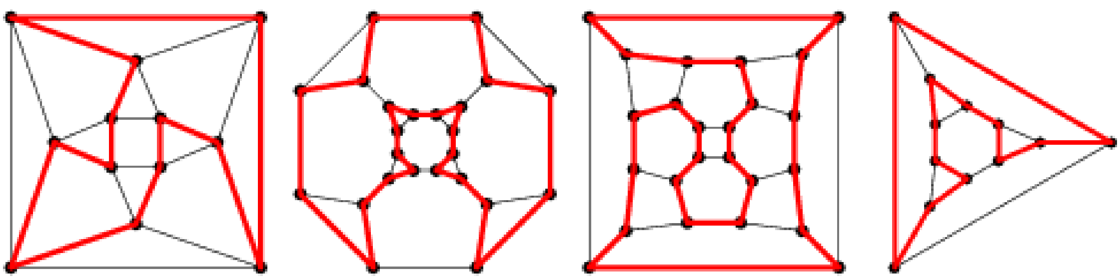
Exercise: If you enjoyed these, you can try the same with some of the following Archimedean solids.



Cuboctahedron | Truncated cube | Truncated octahedral | Truncated tetrahedral

You will first need to stretch one of their larger faces and then squash them to planar graphs:

Solutions:

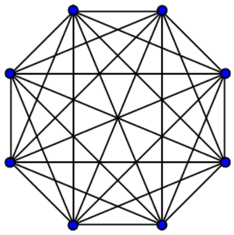


Definition:

A *simple graph* is one where at most one edge joins any two nodes and there are no “loops” (cases where a single edge starts and ends at the same node).

Exercise:

A *complete graph* is a simple graph where every node is connected with every other node by exactly one edge. In the diagram you can see a complete graph with 8 nodes.



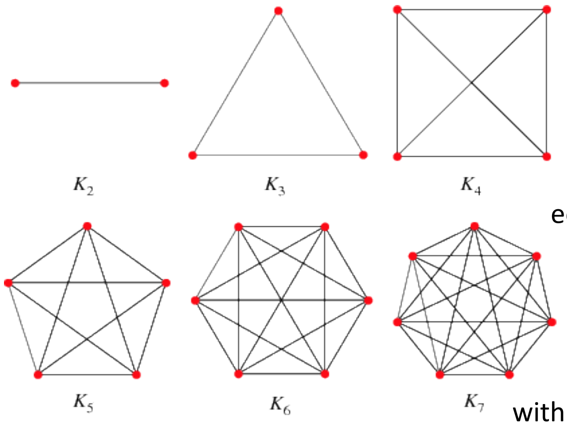
(a) Draw diagrams for simple graphs with:

(i) 2 nodes; (ii) 3 nodes; (iii) 4 nodes; (iv) 5 nodes (v) 6 nodes (vi) 7 nodes

and count their edges. Find a formula for the number of edges of a complete graph with n nodes.

(b) Every complete graph with at least 3 nodes is Hamiltonian. For example, a complete graph with 3 nodes has 1 Hamiltonian cycle, and a complete graph with 4 nodes has 3 Hamiltonian cycles (find them all). Find a formula for the total number of Hamiltonian cycles that can be found in a complete graph with n nodes.

Solution:



Important note: In the diagram, the points not marked by red dotes are not nodes, they just happen to be at the intersection of some edges.

(a) A complete graph with n nodes has $C_2^n = \frac{n(n-1)}{2}$ edges which can be counted in two ways:

- Label the nodes from 1 to n. The first node gets connected with $(n-1)$ nodes, the 2nd node with the remaining $(n-2)$ nodes, the 3rd node with $(n-3)$ nodes etc. In total we have

$$(n-1)+(n-2)+\cdots+2+1=\frac{n(n-1)}{2} \text{ edges.} \quad \text{OR}$$

- Each edge is given by a pair of 2 different nodes. You have n options for the first node and $(n-1)$ for the 2nd node, but the same edge can be obtained if you swap the to nodes, so $\frac{n(n-1)}{2}$ edges.

(b) A complete graph with n nodes will have $\frac{(n-1)!}{2}=\frac{(n-1)(n-2)\cdots 2\cdot 1}{2}$ Hamiltonian cycles. To see this, label your nodes from 1 to n and always start your Hamiltonian cycle from node 1. Since there are edges between any two nodes, all you need to do now is choose the order in which you will pass through the nodes 2,3,…, n. This gives $(n-1)(n-2)\cdots 2\cdot 1=(n-1)!$ Trips.

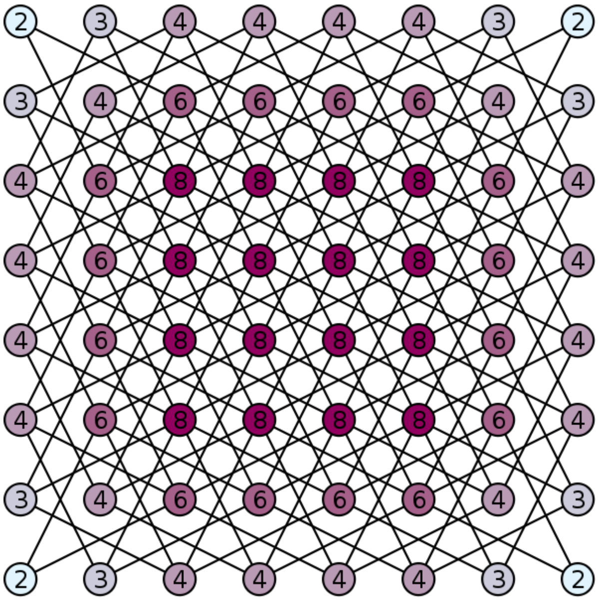
But the same cycle can be travelled on in two different directions, so there are $\frac{(n-1)!}{2}$ cycles.

Application: Knight's Tour and the T.S. Problem

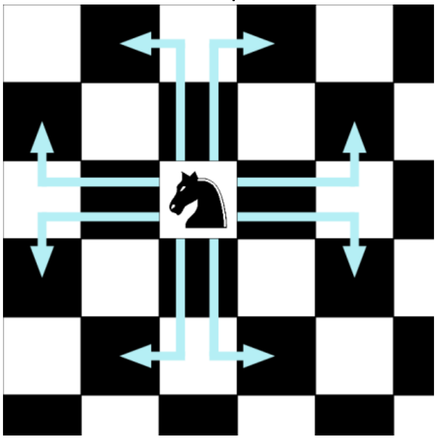
As you can imagine, there are so many different kinds of Hamiltonian path problems. A well-known example is the Knight's Tour in a chessboard:

Making a knight's tour involves moving a knight around a chessboard, landing at each square exactly once (no more nor less often). Knights can only move in "L-shapes", i.e. one square horizontally and two vertically or two squares horizontally and one vertically.

An "open" tour is one that doesn't start where it began; a "closed" tour does. It mightn't seem like this has anything to do with Hamiltonian paths, but think about it: what if we joined each square (using lines) to all squares the knight could move to in one turn? We'd get a graph, with the squares acting as the nodes and the lines as the edges. In fact, it would look like this:



Then, the Knight's Tour game becomes a matter of finding a Hamiltonian path or cycle in this graph. As you can imagine, there are loads of different tours that can be constructed. It is still a very difficult problem, however. It can't really be solved by brute force (listing all possible paths in the graph and picking out the ones that are Hamiltonian) because there are around 4×10^{51} Hamiltonian paths that a knight can take (that's a fairly big number. In fact, if it took you a minute to check each path, it would take about 7.5 billion trillion trillion trillion years in total). There are, however, some algorithms that can produce results. A useful one is Warnsdorff's rule (a "heuristic" algorithm), which tells you that, when choosing your next move, you should always take the option with the lowest number of possible moves. If you ever find that two or more options share this lowest number, you should just pick one of them randomly (there are methods for determining which one to choose, but they're quite complicated).



[Here's](#) a cool Knight's Tour game which can be played online. It's easier to use than the traditional chessboard because it keeps track of the moves you've already made and only permits legal knight moves.

ALGEBRA AND GEOMETRY:
FROM REAL NUMBERS TO COMPLEX NUMBERS AND QUATERNIONS.

- *Look at the Algebra with real numbers as a way to describe movements along a line.*
- *Introduce Complex Numbers as points in the plane, and operations with complex numbers as movements on the plane.*
- *Introduce Quaternion Algebra with the hands-on Quaternion Ball tool.*
- *Perform rotations in 3D using Quaternions*
- *Advanced level: Explain the Rotation Formula using tools from Algebra and Geometry.*

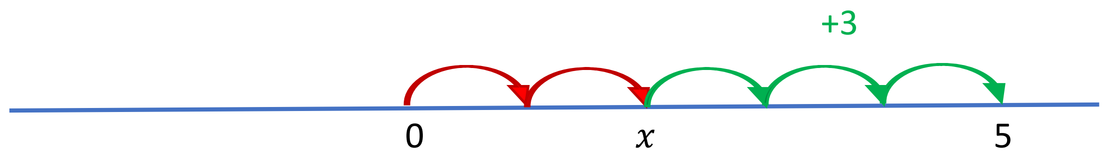
Mathematics is a story spanning thousands of years, with hundreds of characters, both human and mathematical. It is a story too long for anyone to hear the whole of, but many people spend their lives listening. Today we will give a little bit of this story and our main characters will be an Irish mathematician, William Rowan Hamilton, and a new number system called the quaternions. But what are the quaternions and more importantly, why should we care about them? Hamilton came about the idea of quaternions as a way to represent rotations in a three-dimensional space.

NUMBERS ON THE REAL LINE

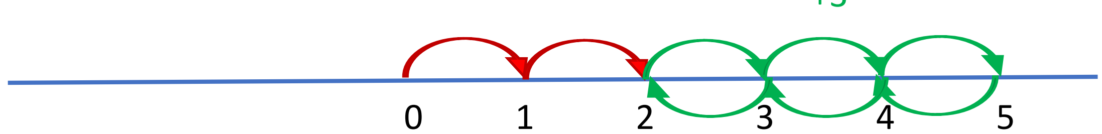
In order to understand these fully, we need to remember how numbers and operations can be thought of in practical (and geometric) terms:

Real numbers = Points on a line Operations with numbers = movements along the line

For example, +3 means we skip three unit steps to the *right*, starting from wherever we are.



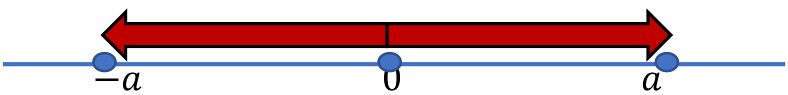
Here we use the symbol x to describe a number that we might not know from the beginning, and whose value we might find out later. If our walk along the number line led us to 5, we write this as $x + 3 = 5$. To find x, we move back 3 spaces: $x = 5 - 3 = 2$.



Thus -3 means moving 3 steps to the *left*. This is the type of thinking that the Persian mathematician *al-Khwārizmī* (780 – 850) described by the Arabic word *al-jabr* (reunion of broken parts), which is the origin of the well-known word *Algebra*.

Algebra uses symbols to describe numbers. This allows us to make general statements like this one (where a stands for any positive number):

$+a$ is a move of length a to the right , while $-a$ is a move of length a to the left.



From the picture we see that $-a$ is the result of the *reflection* of a across 0. Since $-a = (-1) \times a$ we can thus give a geometric meaning to the multiplication by -1:

Geometrically, multiplication by (-1) is the *reflection* across 0.

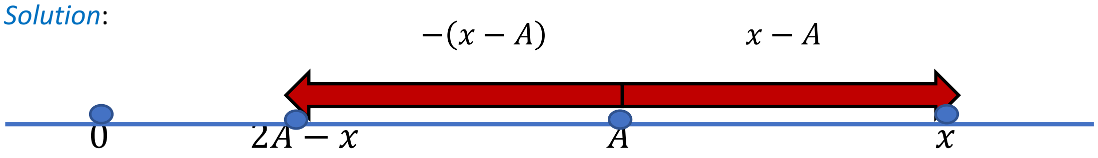
$(-1) \times$ has the effect that every number to the left of 0 gets moved to the right and every number to the right of 0 gets moved to the left.

Exercise: Use reflection to explain why multiplying two negative numbers gives you a positive number.

Solution: As $-b = b \times (-1)$ and $-c = c \times (-1)$, it is enough that multiplication by $(-1) \times (-1)$ means reflecting twice, which takes any number back where it was originally. So $(-1) \times (-1) = 1$.

Exercise: Work out how to reflect around a number other than 0. Let's say that we have a number A on the line and another number x. Write an equation for the reflection of x through A. It should be an expression in x and A.

Solution:



Geometrically, we can move the whole line to the left by A. Then x becomes $x - A$ and A becomes 0. Then reflecting through 0 sends $x - A$ to $-(x - A) = A - x$. Now we need to move the line back to the right by A. This makes $A - x$ into $2A - x$.

COMPLEX NUMBERS

As often happens with stories, we must now skip ahead in time. We come to the second major characters in our story, the complex numbers. Armed with all the numbers, operations and symbols they could put on a line, mathematicians could now solve so many different equations that they even starting looking into impossible ones, like this one:

$$x^2 + 1 = 0$$

which is the same as $x^2 = -1$ or equivalently $x = \sqrt{-1}$ or $-\sqrt{-1}$. But taking the square root of a negative number seemed totally impossible for a long time. The first person who dared mention it was the sixteenth century Italian mathematician Cardano, who called such a number x meaningless, fictitious, and imaginary, while at the same time admitting it would allow him to solve problems like splitting the number 10 into two parts the product of which would be 40.

Exercise: Show that the two numbers that Cardano was seeking, whose sum is 10 and product is 40, would be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ if we allowed for square roots of a negative numbers.

Solution: If we write one of the numbers as x and the other as $10 - x$, we get the equation

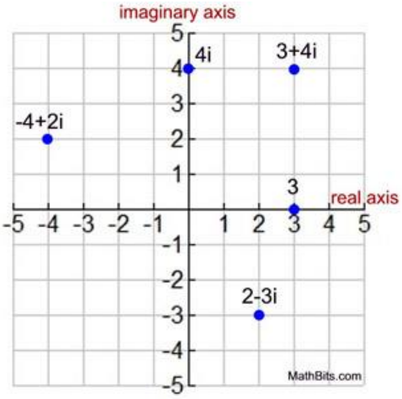
$$x(10-x) = 40$$

which becomes $0 = x^2 - 10x + 40$. After completing the square: $x^2 - 2 \cdot 5 \cdot x + 5^2 = (x+5)^2$ we get $x^2 - 10x + 40 = x^2 - 2 \cdot 5 \cdot x + 25 + 15 = (x+5)^2 + 15$. Thus

$(x+5)^2 + 15 = 0$ hence $(x+5)^2 = -15$ or $x + 5 = \sqrt{-15}$ or $-\sqrt{-15}$ which gives the desired solutions.

Through many years of calling it artificial and impossible, the new number $\sqrt{-1}$ came to be known as $i = \sqrt{-1}$ from *imaginary*. It wasn't the only "imaginary" number either: numbers like $\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2i$, $\sqrt{-9} = \sqrt{9} \cdot \sqrt{-1} = 3i$. Impossible as they seemed, they came to be used more and more in solving all kinds of problems. When combined with the usual numbers and operations, things like $5 + \sqrt{-15} = 5 + i\sqrt{15}$ or more generally, $a + ib$, naturally appear. These kinds of numbers become known as complex numbers.

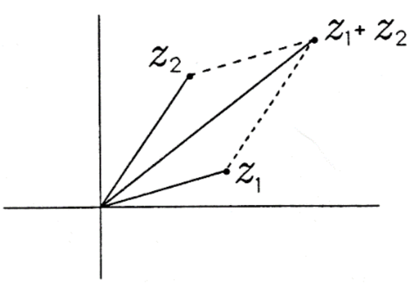
"For well over two centuries after imaginary numbers broke into the domain of mathematics they remained enveloped by a veil of mystery and incredibility until finally they were given a simple geometrical interpretation by two amateur mathematicians: a Norwegian surveyor by the name of Wessel and a Parisian bookkeeper, Robert Argand". According to their interpretation a complex number, as for example $3 + 4i$, may be represented as in the Figure here in which 3 corresponds to the horizontal distance, and 4 to the vertical. (George Gamow, "One, two , three … infinity").



Looking at this geometrically, we now see

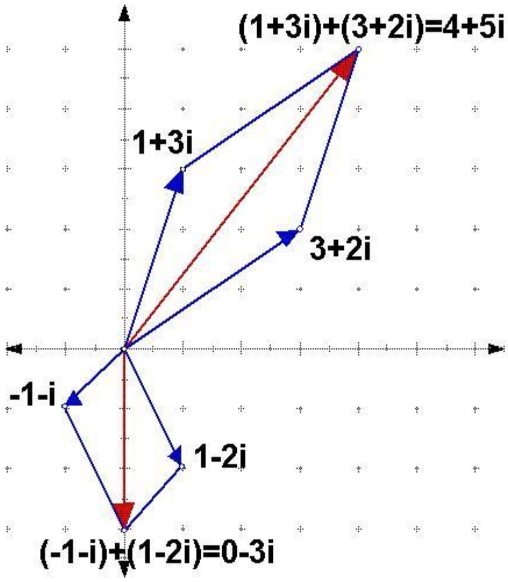
Complex numbers = Points on the plane
and hence we would expect that
Operations with complex numbers = movements on the plane.

Indeed, for every complex number z_2, we can think of $+z_2$ as describing both a distance and a direction of travel on the plane, from wherever we may start. For example in the picture here, the direction of z_2 is North-North-East, and the distance is given by the length of the segment to z_2. Thus $+z_2$ on its own means that we start from the 0 and go to the point z_2, exactly, whereas $z_1 + z_2$ means that we first go to z_1 and from there we travel the direction and distance prescribed by z_2, which should land us at the point marked $z_1 + z_2$ in the diagram.



$+z$ is a move in the direction and the distance of z, from wherever we may start.

Here are some examples:

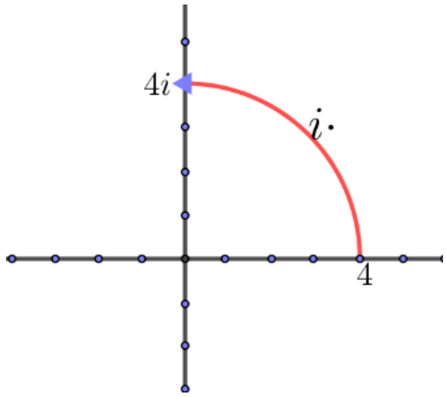


We can always use the sum rule backwards, for example in this diagram we can represent the trip from $3 + 2i$ to $4 + 5i$ as

$$\begin{aligned} &4 + 5i - (3 + 2i) = \\ &= 4 - 3 + 5i - 2i = \\ &= +(1 + 3i). \end{aligned}$$

Products:

Let us now consider products. When we multiply a real number, say 4, representing a point on the horizontal axis, by the imaginary unit i we obtain the purely imaginary number $4i$, which must be plotted on the vertical axis.

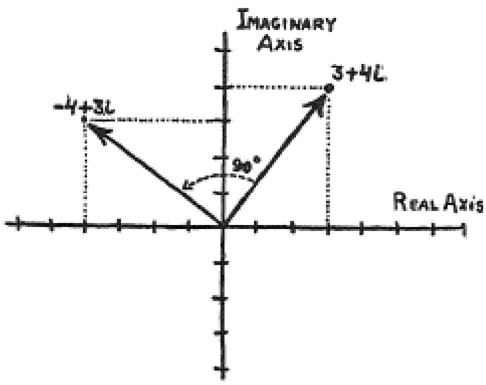


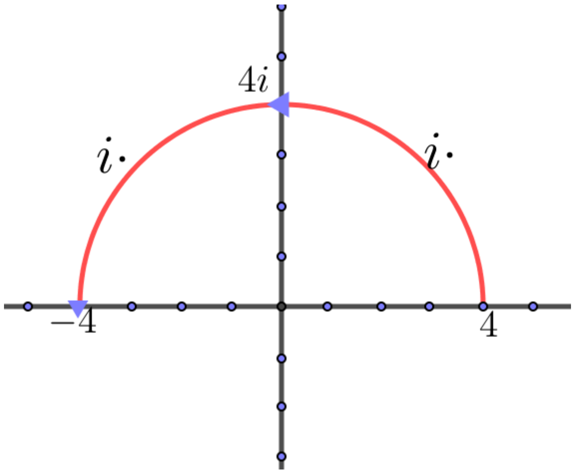
Multiplication by i is the same as a counter-clockwise rotation by a right angle around 0.

Exercise:

Check the position of the point $i \cdot (3 + 4i) = -4 + 3i$ in the figure above. Check that the ray of $(-4 + 3i)$ is perpendicular on the ray of $(3 + 4i)$.

Solution: The two rays have slopes $\frac{3}{4}$ and $-\frac{4}{3}$ which means they're perpendicular.





We can see that rotating by 90° twice corresponds to a rotation by 180°, which is the same as the reflection through the origin:

$$i \cdot i \cdot 4 = -4$$

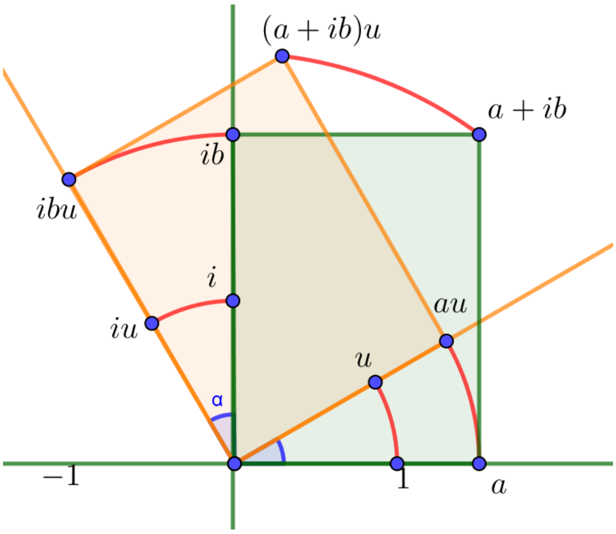
or equivalently $i^2 = -1$. The equation that had baffled mathematicians for centuries now has a very nice geometric interpretation.

(Recall that $i\,\cdot$ is the rotation by $90°$ while $(-1)\cdot$ is the reflection through 0.)

By the same means

Multiplication by $-i$ is the same as a clockwise rotation by a right angle around 0.

Products by complex numbers can describe rotations by all angles:



Let's take a complex number u obtained by rotating the number 1 from the horizontal axis by α degrees. Such a number is written as $u = \cos\alpha + i\sin\alpha$. The picture here should help convince you that multiplication by u is always rotation by α. Indeed, u is just 1 rotated by α by construction. We know that iu is just u rotated by $90°$. Using this, show that iu is also i rotated by α. Since $\cdot\, u$ rotates both axes by α, it follows that it rotates any box like the one in the figure by α, namely $(a+bi)u = au + biu$ where au is just a rotated by α and biu is just ib rotated by α. When we rotate a and ib we rotate the entire box having these as sides, and in particular $(a+bi)u$ is $(a+bi)$ rotated by α.

The Treasure-Hunting Puzzle (from "One, two, three… infinity" by George Gamow):
If you still feel a veil of mystery surrounding imaginary numbers you will probably be able to disperse it by working out how to find some buried treasure using complex numbers.

There was a young and adventurous man who found among his great-grandfather's papers a piece of parchment that revealed the location of a hidden treasure. The instructions read:
"Sail to __________ North latitude and ___________ West longitude where you will find a deserted island. There on a large Meadow stand a lonely oak and a lonely pine. There you will see also an old gallows on which we once hanged traitors. Start from the gallows and walk to the oak counting thy steps. At the oak you must turn right by a right angle and take the same number of steps. Put here a spike in the ground. Now you must return to the gallows and walk to the pine counting the

steps. At the pine you must turn left by a right angle and see that you take the same number of steps, and put another spike into the ground. Dig halfway between the spikes; the treasure is there."

The instructions were quite clear and explicit, so our young man chartered a ship and sailed to the South Seas. He found the island, the field, the oak and the pine, but to his great sorrow the gallows was gone. Too long a time had passed since the document had been written; rain and sun and wind had disintegrated the wood and returned it to the soil, leaving no trace even of the place where it once had stood. Our adventurous young man fell into despair, then in an angry frenzy began to dig at random all over the field. But all his efforts were in vain; the island was too big! So he sailed back with empty hands. And the treasure is probably still there.

A sad story, but what is sadder still is the fact that the fellow might have had the treasure, if only he had known a bit about mathematics, and specifically the use of imaginary numbers. Let us see if we can find the treasure for him, even though it is too late to do him any good.

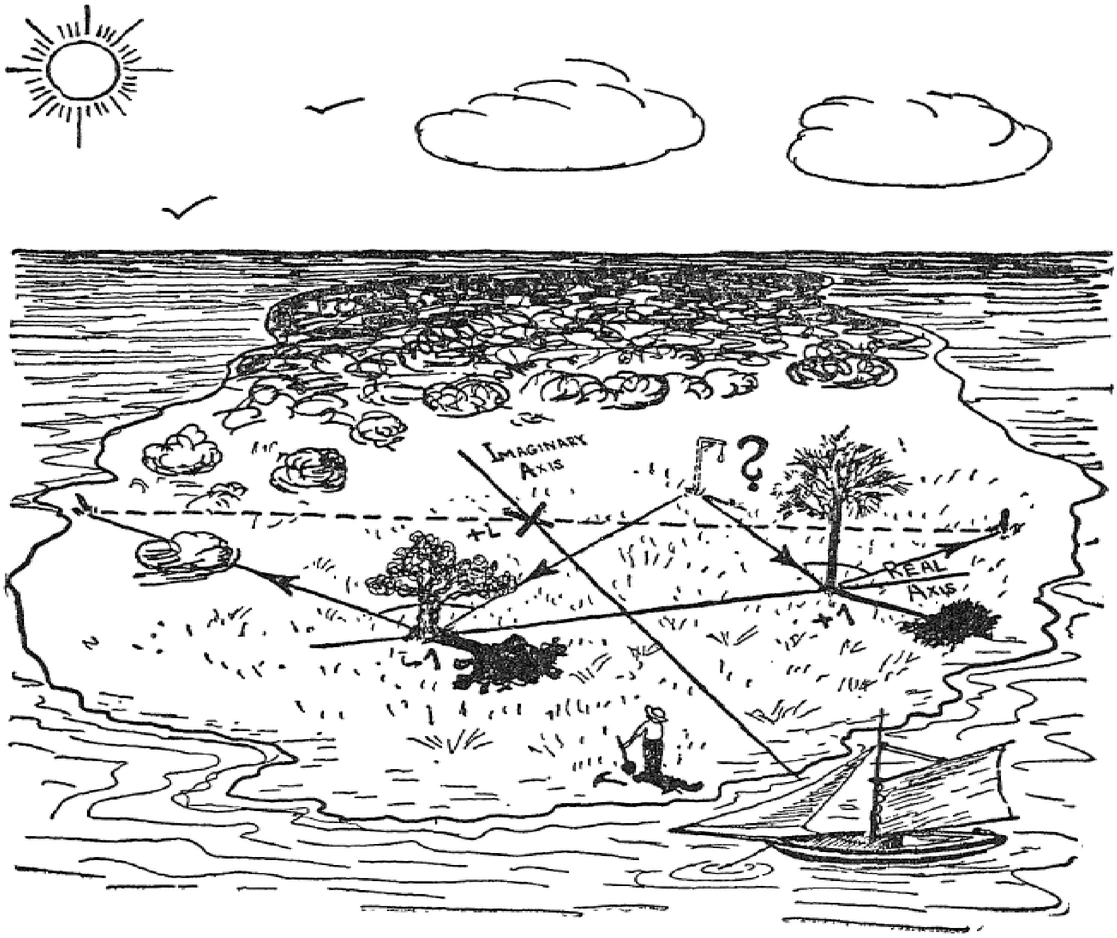


FIGURE 11
Treasure hunt with imaginary numbers.

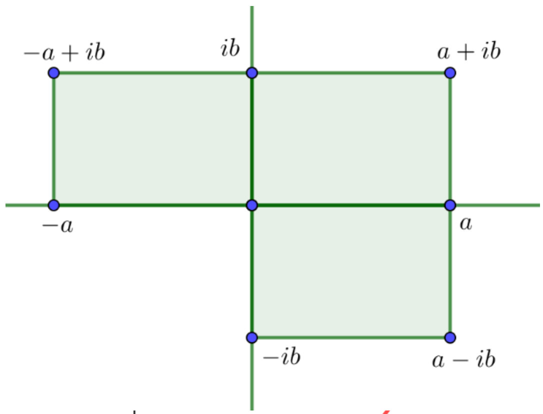
Solution: Consider the island as a plane of complex numbers; draw the real (horizontal) axis through the base of the two trees, and another axis (the imaginary one) at right angles to the first, through a point half way between the trees. Taking $\frac{1}{2}$ the distance between the trees as our unit of length, we can say that the oak is located at the point -1 on the real axis, and the pine at the point $+1$. We do not know where the gallows was so let us denote its hypothetical location by the Greek letter Γ

(capital gamma), which even looks like a gallows. Since the gallows was not necessarily on one of the two axes, Γ must be considered as a complex number: $\Gamma = a + bi$.
Now let us do some calculations remembering the rules of imaginary multiplication as stated above.
If the oak is at -1, the gallows is at Γ, then the walk from the oak to the gallows may be written as $\Gamma - (-1) = \Gamma + 1$. Now to rotate it counter-clockwise by $90°$, we multiply by i and get $i(\Gamma + 1)$.
This describes the trip from the oak to the 1st spike. Since the oak is at point -1, the trip from the origin O to the first spike is: $i(1 + \Gamma) - 1$.
Similarly the walk from the pine to the gallows is written as $\Gamma - 1$. We rotate it clockwise around the pine by multiplying by $-i$ and then calculate the trip from O to the second spike as: $(-i)(\Gamma - 1) - 1 = i(1 - \Gamma) + 1$
Since the treasure is halfway between the spikes, we must now find one half the sum of the two above complex numbers. We get:

$$\frac{1}{2}[\, i(\Gamma + 1) - 1 + i(1 - \Gamma) + 1] = \frac{1}{2}[i\Gamma + i - 1 + i - i\Gamma + 1] = \frac{1}{2}[2i] = i.$$

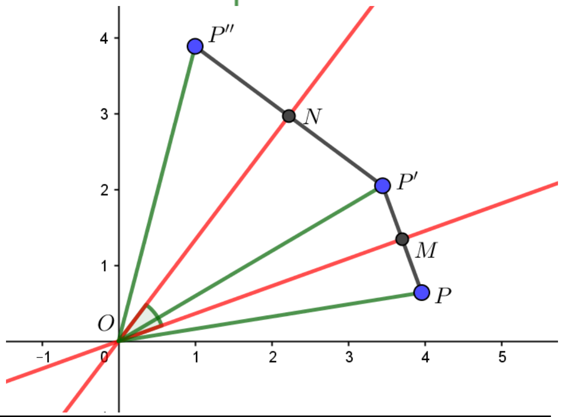
We now see that the unknown position of the gallows denoted by Γ fell out of our calculations somewhere along the way, and that, regardless of where the gallows stood, the treasure must be located at the point $+i$.
And so, if our adventurous young man could have done this simple bit of mathematics, he would not have needed to dig up the entire island, but would have looked for the treasure at the point indicated by the cross in Figure 11, and there would have found the treasure.

Reflections:
Another way to move things in the plane is to reflect across lines. Reflecting a point across the line means drawing a segment through P which is perpendicularly bisected by the line.
For the reflection of $z = a + ib$ across the horizontal and the vertical axis, respectively we have:
Reflection of $z = a + bi$ across the real axis gives $a - ib$. This is called the conjugate of z and is denoted by $\bar{z}$. Reflection across the imaginary axis gives $-a + ib$.



Exercise: Successive Reflections.
Reflections are important in many ways. For example, take two lines l and k meeting at O and with an angle α between them. Reflect a point P through l and then k successively to get P' and then P''.

Show that $|OP''| = |OP'| = |OP|$ and that $\angle P''OP = 2\alpha$ no matter which P you choose.



Two successive reflections across two lines through O amounts to a rotation by double the angle between the two lines

(clockwise or counter-clockwise depending on which line you reflected across first)

Proof: Indeed, reflection means that $\Delta\, POM \equiv \Delta\, P'OM$ and $\Delta\, P'ON \equiv \Delta\, P''ON$ hence $|PO| = |P'O| = |P''O|$ and $\angle\, POM = \angle\, P'OM$ and $\angle\, P'ON = \angle\, P''ON$. Hence by summing up :

$$\angle\, POP'' = \angle\, POP' + \angle\, P'OP'' = 2\angle\, P'OM + 2\angle\, P''ON = 2\,\angle\, MON = 2\,\alpha.$$

Conversely, you can rotate a point P around O by an angle θ , by reflecting it successively across two lines which make an angle of $\theta/2$ at O.

THE QUATERNIONS

- *Introduce the Quaternions as a four-dimensional system.*
- *Describe addition and multiplication of Quaternions, with the use of a geometric visualisation.*
- *Consider some applications of the Quaternions.*

Just like complex numbers numbers $a + bi$ represent points in the plane and are made of a pair of real numbers (a, b), we can represent a point in the 3-dimensional space by a triplet (x, y, z). Hamilton was fascinated by the discovery that multiplication represents rotation in the complex plane, and he wanted to do the same in 3D. The problem of finding an algebra of triples (α, β, γ) to describe the geometry of vectors in three dimensional (3D) space haunted him for at least fifteen years.

“Every morning in the early part of the above-cited month [October 1843], on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, “Well, Papa, can you multiply triplets”? Whereto I was always obliged to reply, with a sad shake of the head: “No, I can only add and subtract them.” W R Hamilton in a letter dated August 5, 1865 to his son A H Hamilton [1].

In 1843, Hamilton found an ingenious way around his problem. The solution famously came to him as he was walking along the Royal Canal in Dublin with his wife on 16th October (now called Hamilton day)- he suddenly realised that the answer lay in numbers with four components instead of three. In his excitement, he promptly used his penknife to carve the solution equations into the side of nearby Broom bridge:

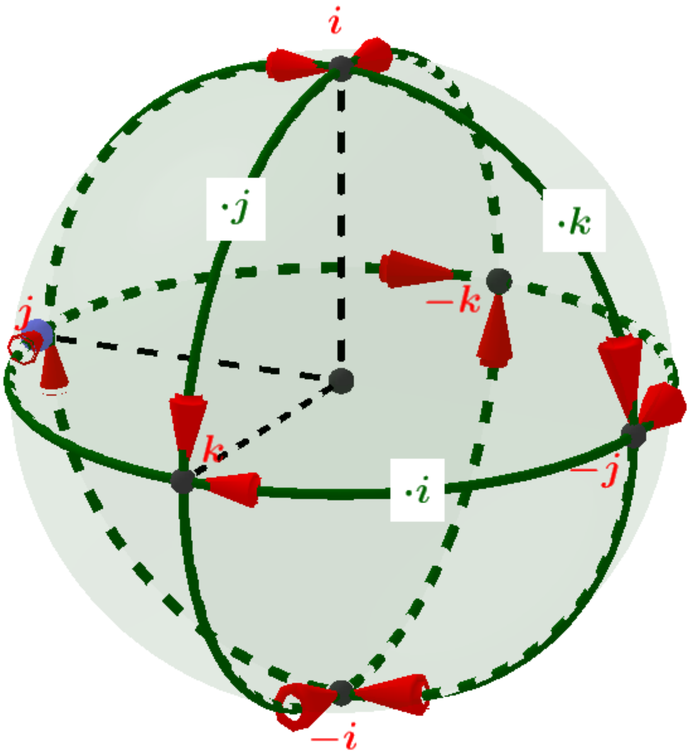
$$i^2 = j^2 = k^2 = ijk = -1$$

Plaque on Broom bridge. Wikimedia Commons.

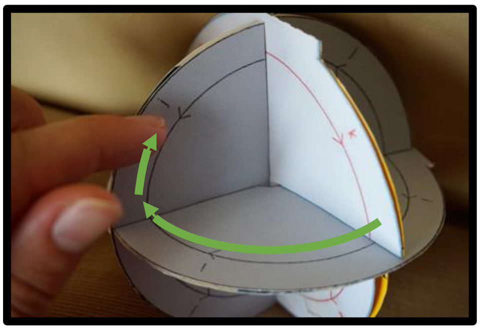
This was the birth of the quaternions, which we represent by $\mathbb{H}$ in honour of Hamilton. Let’s look more closely at what we’ve just written.

First, Hamilton decided that he could have not just 1, but 3 *“imaginary”* axes, each with its own unit: i, j and k are what we call the quaternion units. They form the building blocks of the quaternions. Notice that they all satisfy $i^2 = j^2 = k^2 = -1$, so in fact we now have 4 axes: three for i, j and k, and a fourth one for the real numbers.

To help us better grasp the consequences of the rules above, we need to play with the symbols i, j and k and in particular, to understand their products. Luckily, we have a handy toy available to help us with this task. It is based on the following ball in the 3D quaternion space. As you can see, the axes are marked by the units i, j, k on one side, and $-i, -j, -k$ on the other side. On each of the three coordinate planes marked by circles, multiplication $\cdot i$, $\cdot j$, $\cdot k$ represents a rotation. Indeed, each circle lies on a plane very similar to the complex plane we're met earlier.



If you haven't already done so, please print out and assemble the Quaternion Ball learning tool by clicking here. We will use it to play with quaternion right multiplication. As you can see, the Ball is made of three discs that intersect at right angles, with red circles on each disc. We find that tracing a finger along these circles while carrying out this exercise is helpful. Tracing out a quarter circle in the same direction as the arrow on it corresponds to right the quaternion unit (i, j or k) printed next to the circle. Tracing opposite to the arrow's direction corresponds to the negative of the unit. Tracing out a number of quarter arcs in sequence corresponds to each of the units traced out written down next to each other in the same order: to the right, for example, I trace out i and then $-j$, which matches the multiplication $i(-j) = -ij$. Any other path that takes you from the same starting point to the same finish gives an equal answer: here, I could also have taken $-k$ to get to the same point, so I know now that $-ij = -k$, or $ij = k$. Let everyone in the class please copy the table just below onto a sheet of paper, and using the Quaternion Ball as we've described, fill out all the missing entries.



Note: Multiplication by 1 does not figure on the Quaternion Ball because it represents staying in place: no change.

	$\boldsymbol{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\times$ $\boldsymbol{1}$				
$\boldsymbol{i}$			k	
$\boldsymbol{j}$				
$\boldsymbol{k}$				

Exercise:

Fill in the table with the correct Hamilton products. For each box, its row represents the first number in the product, while its column is the second number.

For example, I've place k in row i and column j because $ij = k$.

Solution:

Once you've filled in the box, you will notice something kinda weird about it. Everybody knows that

$\times$	$\boldsymbol{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{1}$	1	i	j	k
$\boldsymbol{i}$	i	-1	k	$-j$
$\boldsymbol{j}$	j	$-k$	-1	i
$\boldsymbol{k}$	k	j	$-i$	-1

reversing the order of addition doesn't matter, don't they?

$$3 + 4 = 7 = 4 + 3, \quad \text{right?}$$

Similarly for multiplication: $2 \times 5 = 10 = 5 \times 2$. However, we've just said that $jk = i$ and $kj = -i$, which aren't equal. So, order **does** matter when it comes to the quaternions: that's just how they are. That's something to watch out for when dealing with them.

Note, however, that when you multiply just an imaginary and a real, order doesn't matter. For example, $(8)(\boldsymbol{j}) = (\boldsymbol{j})(8)$.

Let's try to play with the rules of algebra now. Whatever you do though, make sure your symbols resist the (great) temptation to swap places: ij is not the same as ji.

Take $ijk = -1$. If we multiply both sides of this equation by $\cdot\ k$ we get $ijkk = (-1)k$. Now, kk is just k^2, which is -1 from the rules above, so $ijkk = ij(-1) = -ij$. Also, -1 times k is equal to $-k$, so we can simplify this to get $-ij = -k$, which is the same as

$$ij = k$$

So far so good. Next, multiply this on the right by j: $ijj = kj$. Again, $jj = j^2 = -1$, so this becomes

$$-i = kj.$$

Continuing like this, we get six equations:

[a] $jk = i$	[b] $ji = -k$	[c] $ki = j$
[d] $kj = -i$	[e] $ij = k$	[f] $ik = -j$

We haven't fully introduced the quaternions yet, however: i, j and k are just the building blocks. A quaternion ($\boldsymbol{q}$) is any number that looks like this:

$$\boldsymbol{q} = (real\ number) \pm (real\ number)\boldsymbol{i} \pm (real\ number)\boldsymbol{j} \pm (real\ number)\boldsymbol{k}$$

So "$2 - 3\boldsymbol{i} + 6\boldsymbol{j} + 2.34\boldsymbol{k}$", "$0 + 1.5\boldsymbol{i} + \pi\boldsymbol{j} - 1001\boldsymbol{k}$" and "$\frac{16}{7} + \frac{1}{7}\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$" are all quaternions. Quaternion addition is simple: just add parts with the same quaternion unit:

$$1 + 2\boldsymbol{i} + 1\boldsymbol{j} + 2\boldsymbol{k} +$$

$$+3 + 4\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k} =$$

$$= 4 + 6\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}.$$

Multiplication then uses Distributivity like this: take "$1 + 2\boldsymbol{j}$" and "$2\boldsymbol{i} + 3\boldsymbol{k}$", for example:

$(1 + 2\boldsymbol{j})(2\boldsymbol{i} + 3\boldsymbol{k}) = 1(2\boldsymbol{i} + 3\boldsymbol{k}) + 2\boldsymbol{j}(2\boldsymbol{i} + 3\boldsymbol{k}) = 2\boldsymbol{i} + 3\boldsymbol{k} + (2\boldsymbol{j})(2\boldsymbol{i}) + (2\boldsymbol{j})(3\boldsymbol{k})$.

We said above that order doesn't matter when you multiply an imaginary by a real, so

$(2\boldsymbol{j})(2\boldsymbol{i}) = (\boldsymbol{j})(2 \times 2)(\boldsymbol{i}) = (\boldsymbol{j})(4)(\boldsymbol{i}) = (4)(\boldsymbol{j})(\boldsymbol{i}) = 4\boldsymbol{ji}$, and so the final answer is

$2\boldsymbol{i} + 3\boldsymbol{k} + 4\boldsymbol{ji} + 6\boldsymbol{jk} = 2\boldsymbol{i} + 3\boldsymbol{k} - 4\boldsymbol{k} + 6\boldsymbol{i} = 8\boldsymbol{i} - 1\boldsymbol{k}$.

Exercise: Try to multiply these out:

1) $(3\boldsymbol{j} + 4\boldsymbol{k})(6\boldsymbol{j} + 8\boldsymbol{k}) =$

2) $(1 + 3\boldsymbol{i} + 10\boldsymbol{j})(1 + 2\boldsymbol{k}) =$

Solutions:

1) $(3\boldsymbol{j} + 4\boldsymbol{k})(6\boldsymbol{j} + 8\boldsymbol{k}) = (3\boldsymbol{j} + 4\boldsymbol{k})(2)(3\boldsymbol{j} + 4\boldsymbol{k}) = 2(3\boldsymbol{j} + 4\boldsymbol{k})(3\boldsymbol{j} + 4\boldsymbol{k}) = 2(9\boldsymbol{j}^2 + 12\boldsymbol{jk} + 12\boldsymbol{kj} + 16\boldsymbol{k}^2) = 2(-9 + 12\boldsymbol{i} - 12\boldsymbol{i} - 16) = 2(-25) = -50$

2) $(1 + 3\boldsymbol{i} + 10\boldsymbol{j})(1 + 2\boldsymbol{k}) = 1 + 2\boldsymbol{k} + 3\boldsymbol{i} + 6\boldsymbol{ik} + 10\boldsymbol{j} + 20\boldsymbol{jk} = 1 + 3\boldsymbol{i} + 20\boldsymbol{i} + 10\boldsymbol{j} - 6\boldsymbol{j} + 2\boldsymbol{k} = 1 + 23\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$

USING THE QUATERNIONS – ROTATIONS IN 3D

- *Explain how a (unit) quaternion can be used to encode a unique rotation.*
- *Look at an application of rotation quaternions.*

Aside from being a fascinating area of study in pure maths, the quaternions have at least one major application in the real world- mathematically representing rotations in three dimensions. There are other ways (like using rotation matrices) to do this, but the algebra is much simpler when using quaternions and certain problems that can arise with matrices are avoided. As a result, rotations are described by quaternions in multiple areas, including video game and movie animation, aircraft and spacecraft attitude control and robotics.

How it's done

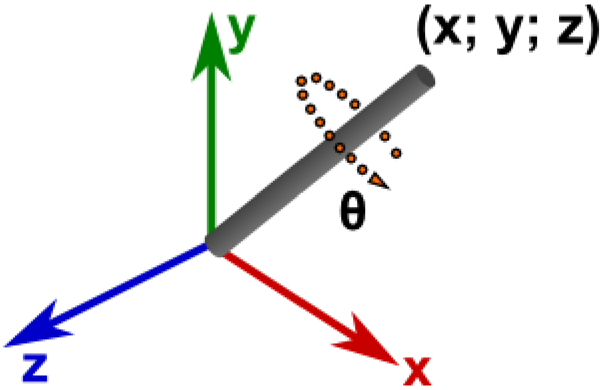
Let's look at how the quaternions can represent rotations in 3-D. Firstly- how do we represent points in three dimensions? You might recall that we said earlier that the quaternions have four components or parts, and so are four-dimensional. How do they represent 3-D then?

The answer is simple- we give every point in 3-D a unique quaternion whose first component is 0.

So, a point in 3-D looks like

$$point\ (\boldsymbol{p}) = 0 + (real\ number)\boldsymbol{i} + (real\ number)\boldsymbol{j} + (real\ number)\boldsymbol{k}$$

Such quaternions are just three-dimensional- they really only have three important components, because 0 doesn't really count here. The three real numbers mentioned can be thought of as the x-, y- and z-coordinates of the point in the 3D space with axes given by i, j, k.



Here, we are going to look at rotations in 3D in terms of two things: an angle that we want to rotate our point by, and the line that we want to rotate the point around. These two things are encoded in a rotation quaternion $\boldsymbol{q}$. This quaternion may have a non-zero first component. This component encodes the angle that we want to rotate by. The $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ components then tell us all about the line we're rotating around. Specifically, if we treat these last three components as a point in 3-D and join that point to the origin (the point $(0,0,0)$), the joining line is the line we're rotating around (the "axis of rotation"). It is determined by the axis quaternion

$\boldsymbol{n} = \ x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ with $\boldsymbol{x^2} + \boldsymbol{y^2} + \boldsymbol{z^2} = \boldsymbol{1}$.

To rotate the point $\boldsymbol{p}$ from above, we just do the following calculation:

$$rotation\ of\ point\ p\ by\ an\ angle\ \theta\ around\ the\ line\ in\ the\ direction\ (x, y, z) = p' = \boldsymbol{q}p\boldsymbol{q}^{-1},$$

where $\boldsymbol{q} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\boldsymbol{n} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)[x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}]$

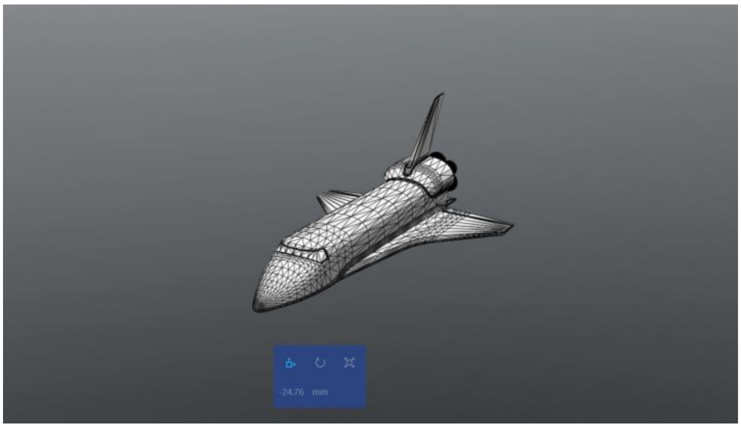
and $\boldsymbol{q}^{-1} = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\boldsymbol{n} = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)[x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}]$.

You can check that $\boldsymbol{q}\boldsymbol{q}^{-1} = 1$, but $\boldsymbol{q}p\boldsymbol{q}^{-1} \neq \ \boldsymbol{q}\boldsymbol{q}^{-1}p = p$! Remember that the order matters in the product of quaternions (it is not commutative)!

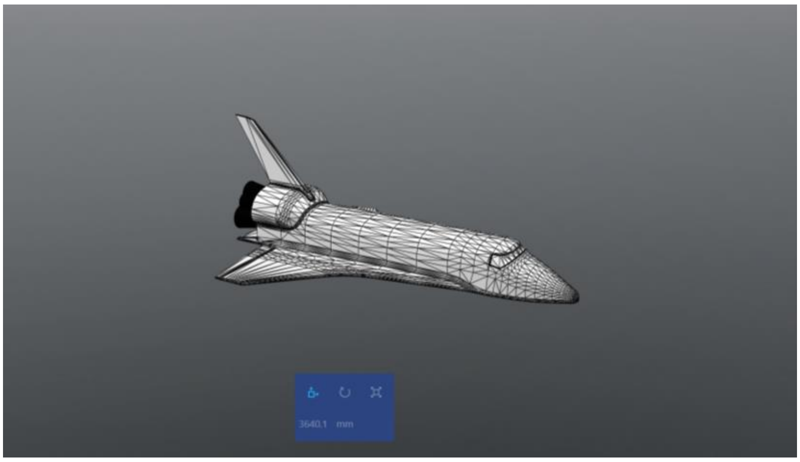
For those who are curious, we will explain latter why this is the right formula for rotations. For the time being, let us just get more comfortable with it by looking at an example.

Quaternions in Spaaace!

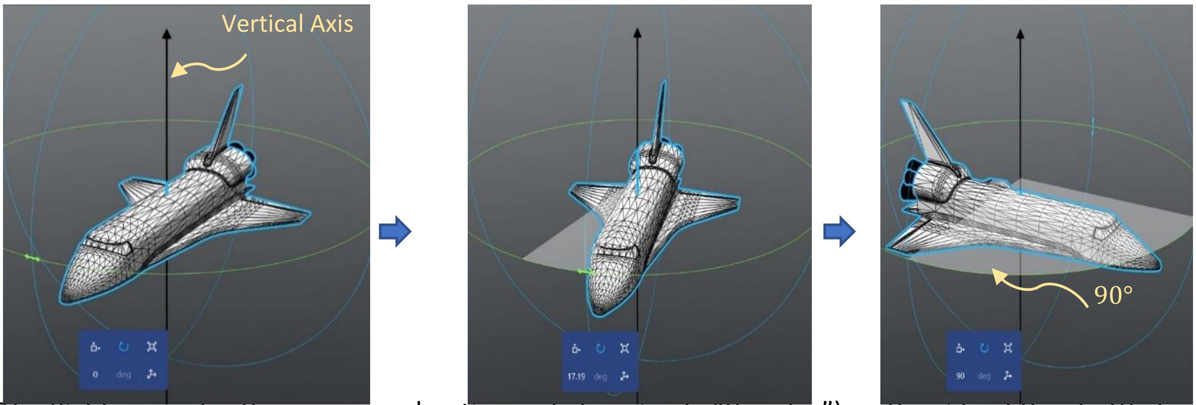
As we explained before, quaternions are used for controlling spacecraft attitude, so we're now going to look at how exactly this works. By the way, attitude is just a terms used in astronautics and other areas that means orientation in 3-D. Now, let's think about a space shuttle in orbit above earth:



Example: Imagine the astronauts are having a tanning competition on board and want to turn the shuttle by $90°$ to its left so that it faces the sun, like this:



How does the shuttle carry out this command? To be mathematical, we say that we want to rotate the shuttle by $90°$ around a vertical axis:

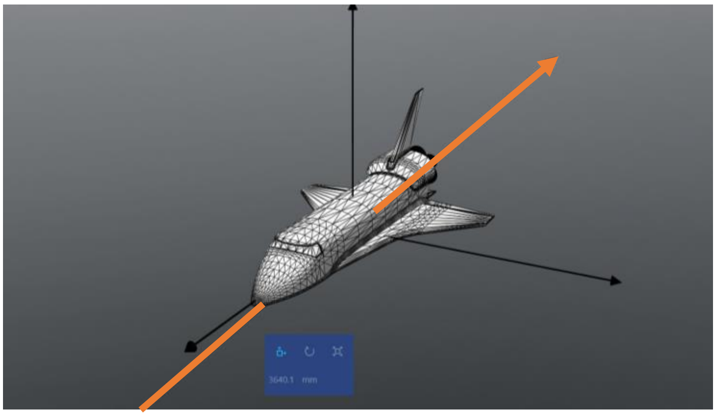


The flight computer then commands a tiny rocket engine (a "thruster") on the side of the shuttle to turn on, causing the shuttle to spin $90°$ to its left. The thruster only fires for a short period of time (or else the shuttle will continue rotating indefinitely), before turning off. Success!

Solution: From the point of view of the quaternion mathematics, we want to find the quaternion $\boldsymbol{q}$ that encodes this rotation. First of all, we need to decide on x, y and z in the expression we looked at in the previous section. x, y and z should be the coordinates of the points that are joined to the origin by the axis of rotation, so we'll pick $x = 0, y = 0, z = 1$ (i.e. the point $(0,0,1)$ here. You could very well ask why we don't choose $(0,0,-1)$ instead- after all, this would also result in a vertical axis, wouldn't it? The difference, however, is that this axis would point vertically downwards instead of upwards. The maths of quaternions dictates that the rotation is anticlockwise if you are looking along the axis with its arrow pointing towards you. So, if you chose $(0,0,-1)$ and inverted the axis of rotation, the rotation would take place in the opposite direction instead. We are now ready to write out $\boldsymbol{q}$:

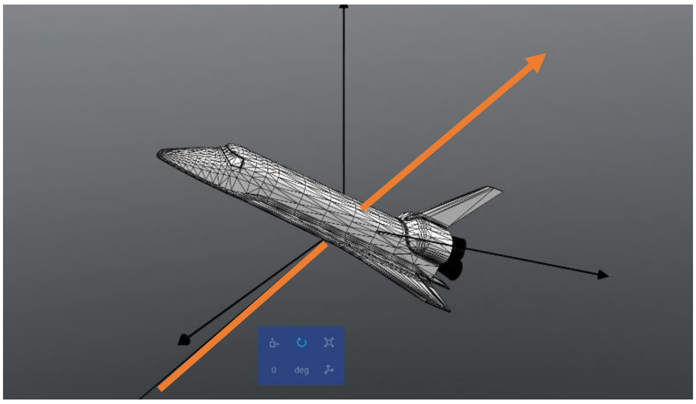
$$\boldsymbol{q} = \cos\left(\frac{90^\circ}{2}\right) + \sin\left(\frac{90^\circ}{2}\right)[(0)\boldsymbol{i} + (0)\boldsymbol{j} + (1)\boldsymbol{k}] = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\boldsymbol{k}$$

Exercise: Let's try something a bit trickier this time. We will try to rotate clockwise by $45°$ around the following axis this time (maybe they've seen aliens or a space duck off to their right, or something):



This time, we'll use $\left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ when we talk about the axis. Can you fill in the boxes in the rotation quaternion?

$$\boldsymbol{q} = \boxed{} + \boxed{}\,\boldsymbol{i} + \boxed{}\,\boldsymbol{j} + \boxed{}\,\boldsymbol{k}$$



*Solution***:**

We are rotating clockwise this time, so we must fill in $-45°$ (because we are rotating by $45°$ in the opposite direction to usual). So,

$$\boldsymbol{q} = \cos\left(\frac{45^\circ}{2}\right) + \sin\left(\frac{45^\circ}{2}\right)\left[0\boldsymbol{i} + \frac{\sqrt{2}}{2}\boldsymbol{j} + \frac{\sqrt{2}}{2}\boldsymbol{k}\right]$$

$$= 0.92 + 0.27\boldsymbol{j} + 0.27\boldsymbol{k}$$

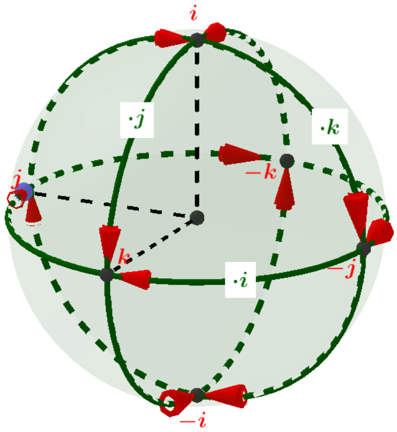
Once the flight computer has figured this out, it commands a number of thrusters to fire in just the right way for this to happen.

This is just one of countless examples of where rotation quaternions are used. Now, the rotation formula may seem to have fallen at us out of the blue sky, so for the curios among you, we will attempt to explain:

GEOMETRY AND QUATERNIONS: EXPLAINING THE ROTATION FORMULA.

Remember that Hamilton discovered quaternions on the way to explaining how to use multiplication to rotate objects in 3D. We will work in the 3D space whose 3 axes have units i, j and k.

Recall the Quaternion Ball we have played with earlier. First off, we should notice that although each on their own circle, the multiplications $\cdot\, \boldsymbol{i}$, $\cdot\, \boldsymbol{j}$ and $\cdot\, \boldsymbol{k}$ represent rotations by $90°$, none of them represents a rotation on the entire 3D space.



Indeed, $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, which is a plain number and does not even exist the 3D-space of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ (whose points are all of the form $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$).

Thus trying to "rotate" i by i would land us in a 4th dimension altogether! Depending on your viewpoint, this is either very exciting or disappointing:

Product of quaternions is not rotation in 3D space.

But let's not despair. The product of quaternions has all kinds of nice geometric connections which we will explore.

Exercise: Geometric Exploration: Here's a nice transformation in 3D:

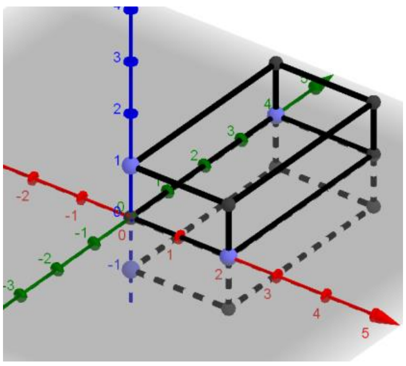
a) Calculate the products iji, iki and iii. Now take a number $ai + bj + ck$ and surround it by i and i, like this: $i(ai + bj + ck)i$. Plot your results in the 3D space. What do you notice? Can you describe this operation as a geometric transformation (movement) of the point $p = ai + bj + ck$?

b) Try the same problem replacing $i$by j: Calculate $jij, jjj, jkj, j(ai + bj + ck)j$. What do you notice?

Solution: $iii = -i, iji = ki = j, iki = ij = k,$

$$i(ai + bj + ck)i = -aiii - biji - ciki = -ai + bj + ck$$

which is the reflection across the plane of axes j and k. This is the plane perpendicular to i.

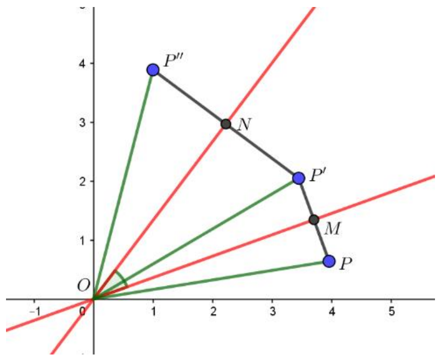


Similarly, surrounding a point p by j-s is the same as reflecting it across the plane perpendicular to the j axis.

We can generalize this by using any quaternion $u = xi + yj + zk$ which lies on the sphere (namely, such that $x^2 + y^2 + z^2 = 1$.)

The product upu = the reflection of p across the plane through 0 and perpendicular to u.

This Claim is a little more difficult to prove in general so we leave it for the advanced section. Let's assume for the moment that this is true.



Suppose we want to rotate p by an angle θ around an axis. Remember how we showed that two successive reflections across lines at an angle $\frac{\theta}{2}$ amount to a rotation by angle θ in the plane? We can do the same thing in 3D, if we replace the lines of reflection with planes. Now all that's left for us to do is to choose two planes containing the axis, and making an angle of $\frac{\theta}{2}$ between them. We will reflect p successively through the two planes. When viewed from straight on top of the rotation axis, the picture looks exactly like the plane one here (Because the planes of reflection are perpendicular to our line of viewing, we see them as lines. We see the rotation axis as a point!) The result is the rotation of p by an angle θ around our axis.

Now let's write this in algebra. Suppose that u and v are vectors in 3D perpendicular to our two planes above. Then the two reflections written successively are $p \to upu \to vupuv$.

$vupuv$ = two successive reflections of p across the planes through O perpendicular on u and v.
= rotation by angle $2\alpha = \theta$ around the axis n which is perpendicular on both u and v.

We claim that

$$\boldsymbol{vu} = \boldsymbol{q} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)[x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}]$$

like in the rotation formula we used before, while $\boldsymbol{uv} = \boldsymbol{q}^{-1} = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)[x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}]$.

This explains why the formula indeed produces a rotation in the 3D i, j, k- space!

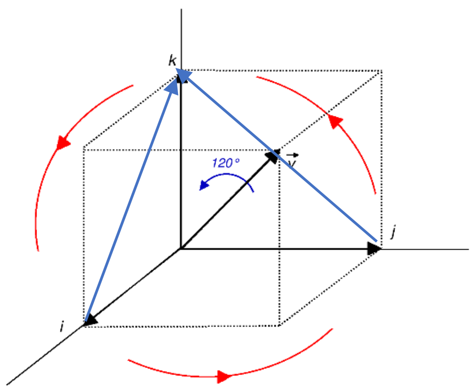
Exercise: Consider the quaternions

$u = \frac{1}{\sqrt{2}}(i - k)$ and $v = \frac{1}{\sqrt{2}}(j - k)$. The directions of these vectors are indicated by the blue arrows in the diagram.

Find the result of the transformation

$$p \to upu \to vupuv.$$

in the cases $p = i$, $p = j$ and $p = k$. Conclude that this transformation is the rotation by $120°$ around the diagonal of the cube – as shown in the picture.



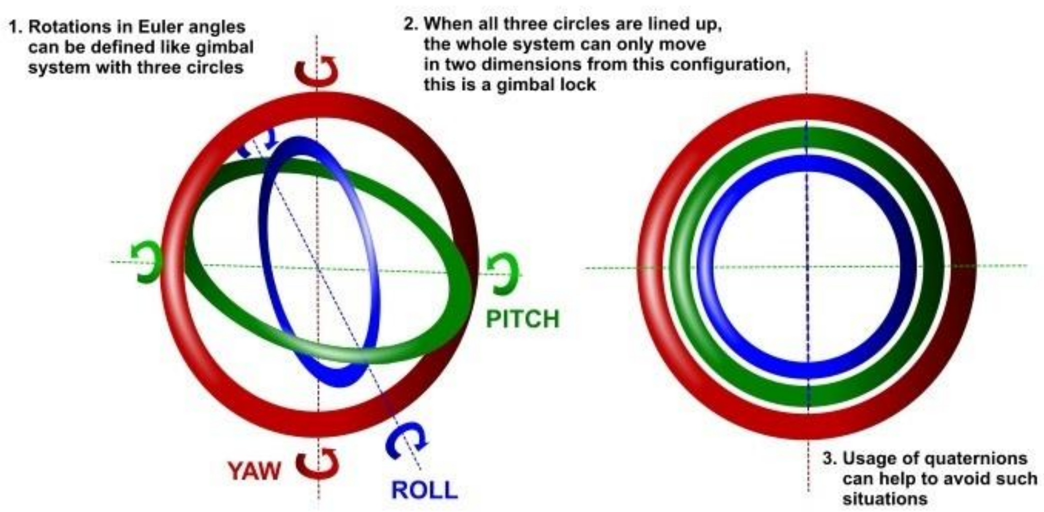
Solution:

$uv = \frac{1}{2}(i-k)(j-k) = \frac{1}{2}(-1+i+j+k)$ and $vu = \frac{1}{2}(j-k)(i-k) = \frac{1}{2}(-1-i-j-k)$.

Then $vuiuv = \frac{1}{4}(-1-i-j-k)i(-1+i+j+k) = \frac{1}{4}(-1-i-j-k)(-i-1+k-j) =$

$$= \frac{1}{4}(i+1-k+j-1+i+j+k-k+j-i-1+j+k+1-i) = j$$

And similarly $vujuv = k$ and $vukuv = i$. Thus this transformation sends i to j and j to k and k to i which is what the rotation by $120°$ around the diagonal of the cube does.



https://www.quora.com/What-are-some-real-life-applications-of-complex-numbers-in-engineering-and-practical-life

Hamilton was delighted with his discovery of Quaternions, and anticipated many future uses. Indeed, Quaternions have found uses varying from the positioning of planes and of space shuttles to computer graphics. Still, Hamilton's great contribution to Mathematics does not consist so much in the Quaternions themselves, as in setting an example by relaxing laws and crossing boundaries of thought, from solving 3D problems by introducing 4 dimensions to the new non-commutative Algebra he designed. Today, non-commutative algebraic and geometric methods continue to play a large role in Mathematics and Theoretical Physics.

ADVANCED SECTION: FOR MATHS CLUBS, MATHS CIRCLES OR MATHS PROJECTS

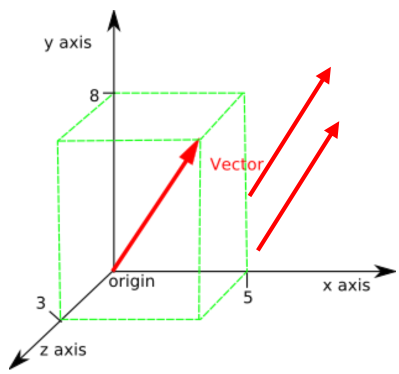
The remaining section is for the truly curious minds among you. There are many beautiful connections between the 3D quaternion algebra and geometry.

Throughout this section, we will regard u and v as vectors in the 3D space:

A vector is an arrow with a direction and a length. We can move it around provided we don't change these two feature.

For example, the red vector in this picture has length

$$|v| = \sqrt{3^2 + 5^2 + 8^2}.$$



You can use a similar formula to calculate the length of a vector in 4 dimensions, but we can only see 3 dimensions in this universe, so we can't accurately draw a 4D vector.

Exercise: Multiply two quaternions $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$.

Use $i^2 = j^2 = k^2 = -1,\ \ ij = -ji = k,\ \ jk = -kj = i,\ \ ki = -ik = j$ as before to write the formula for uv in the form $\boldsymbol{uv} = \boxed{} + \boxed{}\ \boldsymbol{i} + \boxed{}\ \boldsymbol{j} + \boxed{}\ \boldsymbol{k}$

After doing this thoroughly, you will get this formula:

$$uv = -(u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_2 v_3 - u_3 v_2)\ \boldsymbol{i} + (u_3 v_1 - u_1 v_3)\ \boldsymbol{j} + (u_1 v_2 - u_2 v_1)\ \boldsymbol{k}.$$

Now here is a wonderful property of the quaternion product that Hamilton has long sought:

$$|uv| = |u| \cdot |v|$$

The length of a product is the product of lengths.

Check this formula using the coordinates.

Solution: After squaring both sides, this becomes:

$$(u_1 v_1 + u_2 v_2 + u_3 v_3)^2 + (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$
$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)\ .$$

From now on we will take u and v to be points on the quaternion sphere ($|u| = 1 = |v|$), just so as to make our lives simpler. Then the formula above reads:

$$(u_1 v_1 + u_2 v_2 + u_3 v_3)^2 + (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 = 1.$$

As squares are always positive, each square above must be a number between 0 and 1, so we can always write

$$\cos\alpha = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad \text{and}$$

$$\sin\alpha = \sqrt{(u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2}$$

for some angle $\alpha < 90°$. If we do this, then the formula above reduces to $\cos^2\alpha + \sin^2\alpha = 1$, and $\sin\alpha$ is the length of the horribly long 3D vector above

$$\sin\alpha = |\ (u_2 v_3 - u_3 v_2)\ \boldsymbol{i} + (u_3 v_1 - u_1 v_3)\ \boldsymbol{j} + (u_1 v_2 - u_2 v_1)\ \boldsymbol{k}|$$

Divide this vector by its length to get a point $\boldsymbol{n}$ on the quaternion sphere (because $|\boldsymbol{n}| = 1$).

The formula for uv above becomes

$$uv = -\cos\alpha + \sin\alpha \cdot \boldsymbol{n}$$

This looks a lot like to formula for the rotation quaternion $\boldsymbol{q}^{-1}$ which we used when explaining 3D rotations around axes! Also, check that

$$vu = -\cos\alpha - \sin\alpha \cdot \boldsymbol{n}$$

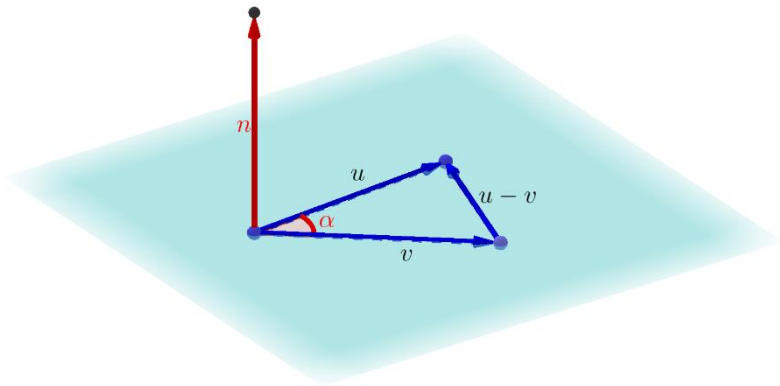
So this is where the quaternion rotation formula comes from: $\boldsymbol{q} = -\boldsymbol{vu}$ and $\boldsymbol{q}^{-1} = \boldsymbol{uv}$ for some vectors u and v in 3D space! This explains why writing rotation in 3D requires a 4th dimension: because this is what happens when we multiply two 3D quaternions u and v!

So we managed to write uv in the form $uv = -\cos\alpha + \sin\alpha \cdot \boldsymbol{n}$, but do you think that α and n are just random numbers? No way!

Claim**:** The geometric significance of the quaternion products of 3D unit vectors:

$$uv = -\cos\alpha + \sin\alpha \cdot \boldsymbol{n}$$

Where α is the angle between u and v , while $\boldsymbol{n}$ is the unit vector perpendicular to both u and v.



We'll prove the Claim through an

Exercise:

(a) Apply the cosine formula in the triangle formed by the quaternions u and v with $u - v$, to check

$$\cos\alpha = u_1v_1 + u_2v_2 + u_3v_3.$$

(b) Apply the formula (a) for the cosine of an angle between two vectors to show that n is perpendicular to both u and v.

Hint: You need to show that $\cos\,(angle\ between\ n\ and\ u) = 0$ and similarly for n and v

You can substitute $(u_2v_3 - u_3v_2)\,\boldsymbol{i} + (u_3v_1 - u_1v_3)\,\boldsymbol{j} + (u_1v_2 - u_2v_1)\,\boldsymbol{k}$ for $\boldsymbol{n}$**,** since n is just a rescaling of the former vector.

Solution: (a) $\cos\alpha = \frac{|u|^2+|v|^2-|u-v|^2}{2|u||v|}$ and since $|u| = |v| = 1$

$$= \frac{1}{2}[u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2 - (u_3 - v_3)^2]$$

$$= u_1v_1 + u_2v_2 + u_3v_3.$$

This confirms that the angle between u and v is the same as α (at least $\pm$).

(b) Using the formula for cosine from (a), we calculate

$$(u_2v_3 - u_3v_2)\,u_1 + (u_3v_1 - u_1v_3)\,u_2 + (u_1v_2 - u_2v_1)\,u_3 = 0$$

(after simplification) and similarly for v.

Let p be any 3D vector. It remains to prove the ***Reflection Formula**:*

upu = the reflection of p across the plane through 0 which is perpendicular to u.

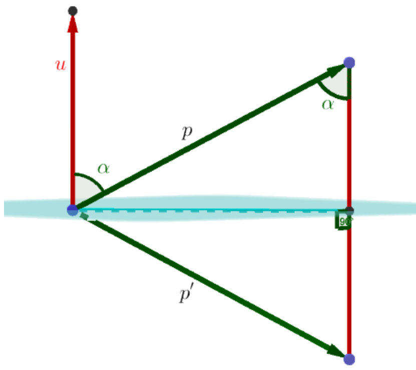
Exercise:

In this exercise, you can work with a point p on the Quaternion Sphere so $|p| = 1$.

(a) Using the diagram here, show that the reflection of p across the plane perpendicular on u is the vector

$$p' = p - 2\cos\alpha \cdot u$$

(b) $up + pu = -2\cos\alpha$ and $upu = p - 2\cos\alpha \cdot u = p'$



Solution: The trip to p' can be completed by walking along p and then down along the red path which is parallel to u. Half of the red path is in a right angled triangle with angle α and hypotenuse $|p| = 1$, so its length is $\cos\alpha$. Thus the entire red paths downwards is in the direction $-u$, and has length $2\cos\alpha$.

(b) We've seen before that we can write $up = -\cos\alpha + \sin\alpha \cdot \boldsymbol{n}$ and $pu = -\cos\alpha - \sin\alpha \cdot \boldsymbol{n}$. Summing up gives $up + pu = -2\cos\alpha$ or equivalently $up = -pu - 2\cos\alpha$.

Substituting this in $upu = (-pu - 2\cos\alpha)u = -pu^2 - 2\cos\alpha \cdot u$.

To recap:

upu = the reflection of p across the plane through O perpendicular on u.
$vupuv$ = two successive reflections of p across the planes through O perpendicular on u and v.
= rotation by angle 2α around the axis n which is perpendicular on both u and v.
And $uv = -\cos\alpha + \sin\alpha \cdot \boldsymbol{n}$, $vu = -\cos\alpha - \sin\alpha \cdot \boldsymbol{n}$
Putting these together we get the formula for rotation by $\theta = 2\alpha$ around $\boldsymbol{n}$. (with a $-$ sign that has to do with whether we take a clockwise or counter-clockwise rotation.

This lesson plan was prepared by Thomas Sheerin, Anca Mustata and Jacob Bennett-Woolf for the Irish Mathematical Trust programme *Mathematicians in Our Lives*. We have used a large number or resources. We most warmly recommend in particular:

- "One Two Three . . . Infinity: Facts and Speculations of Science" - book by: George Gamow
- Wolfram MathWorld http://mathworld.wolfram.com/
- "Men of Mathematics. The Lives and Achievements of the Great Mathematicians from Zeno to Poincare" – book by E.T. Bell
- https://www.forbes.com/sites/chadorzel/2015/08/13/what-has-quantum-mechanics-ever-done-for-us/#1c65408e4046

- https://www.slideshare.net/edzontatualia/refraction-of-light-45755877